Quantitative Abstraction Refinement

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Joint work with:

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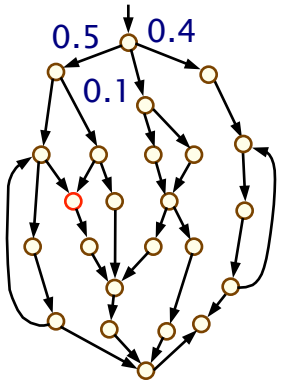
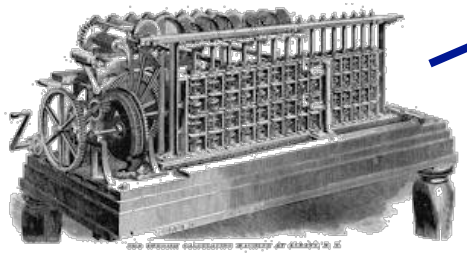
PEPA Club, University of Edinburgh, November 2009

Probabilistic model checking

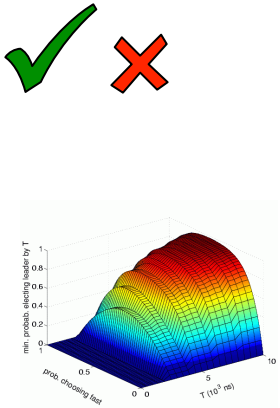
Probabilistic model
e.g. Markov chain



Probabilistic model checker
e.g. PRISM

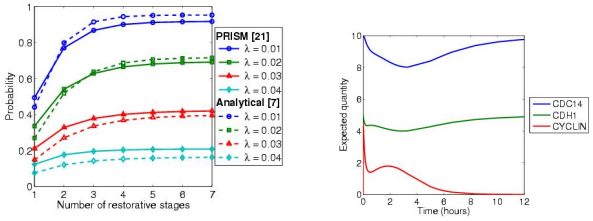


Result

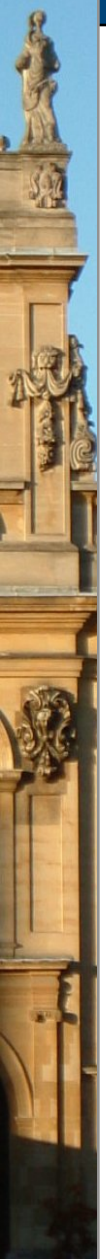


$P_{<0.01}$ [F error]

Probabilistic temporal logic specification
e.g. PCTL, CSL, LTL



Quantitative results



Overview

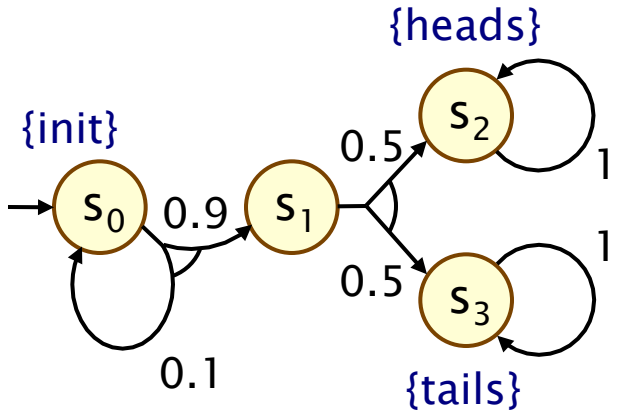
- Probabilistic model checking
  - discrete-time Markov chains (DTMCs)
  - Markov decision processes (MDPs)
  - probabilistic reachability
- Abstraction for probabilistic models
  - abstractions of DTMCs (MDPs)
  - abstractions of MDPs (two-player stochastic games)
- Quantitative abstraction refinement
  - abstraction-refinement loop
  - verification of probabilistic software, real-time systems
- Current/future work & Conclusions

Probabilistic models

- Discrete-time Markov chains (DTMCs)
  - discrete states, discrete probability distributions
- Markov decision processes (MDPs)
  - discrete states, probability and nondeterminism
  - uses: concurrency, under-specification, abstraction
  - models: randomised distributed algorithms, randomised communication protocols, security protocols, …
- Continuous-time Markov chains (CTMCs)
  - discrete states, exponentially distributed delays
  - performance modelling, biological reaction systems, …

Discrete-time Markov chains (DTMCs)

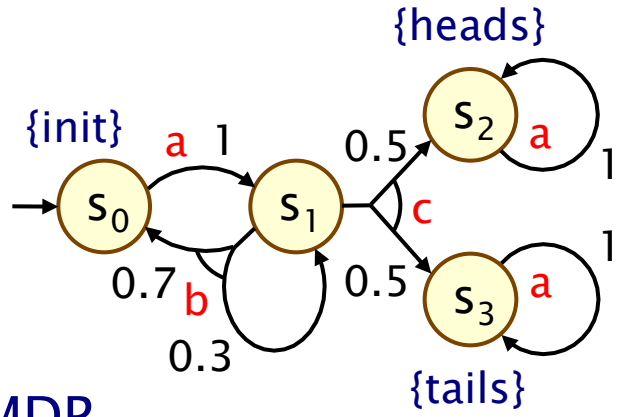
- Model fully probabilistic behaviour
  - state-transition systems augmented with probabilistic choice
- Transition probability matrix
  - over state space S
  - P : S x S → [0,1]



- Paths through a DTMC
  - finite/infinite state sequences
  - Path(s) = set of all (infinite) paths from state s
  - define probability space Pr_s over Path(s)
- Probabilistic reachability (for a set of goal states $F \subseteq S$)
  - probability of reaching F from state s
  - $p_s(F) = Pr_s \{ s_0 s_1 s_2 s_3 \ldots \in Path(s) \mid s_i \in F \text{ for some } i \}$
  - reduces to solution of a linear equation system

Markov decision processes (MDPs)

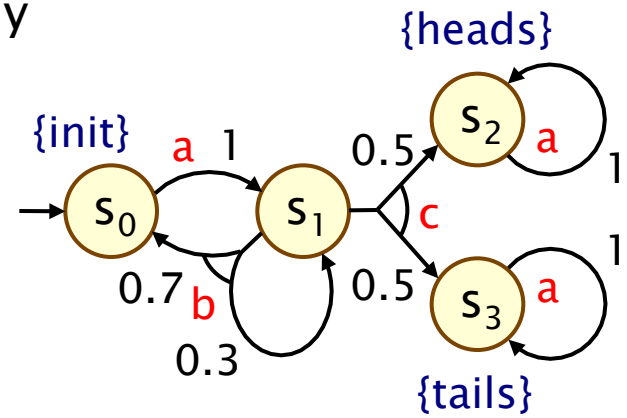
- Model nondeterministic as well as probabilistic behaviour
  - extension of discrete-time Markov chains
  - nondeterministic choice between probability distributions
- Transition probability function
  - Steps : $S \rightarrow 2^{Act \times Dist(S)}$
  - maps states to non-empty sets of action-probability distribution pairs



- A (finite or infinite) path through an MDP
  - is a sequence of (connected) states
  - represents an execution of the system
  - resolves both the probabilistic and nondeterministic choices

Strategies of MDPs

- A strategy (aka. "adversary", "scheduler", …) of an MDP
  - is a resolution of nondeterminism only
  - is (formally) a mapping from finite paths to distributions



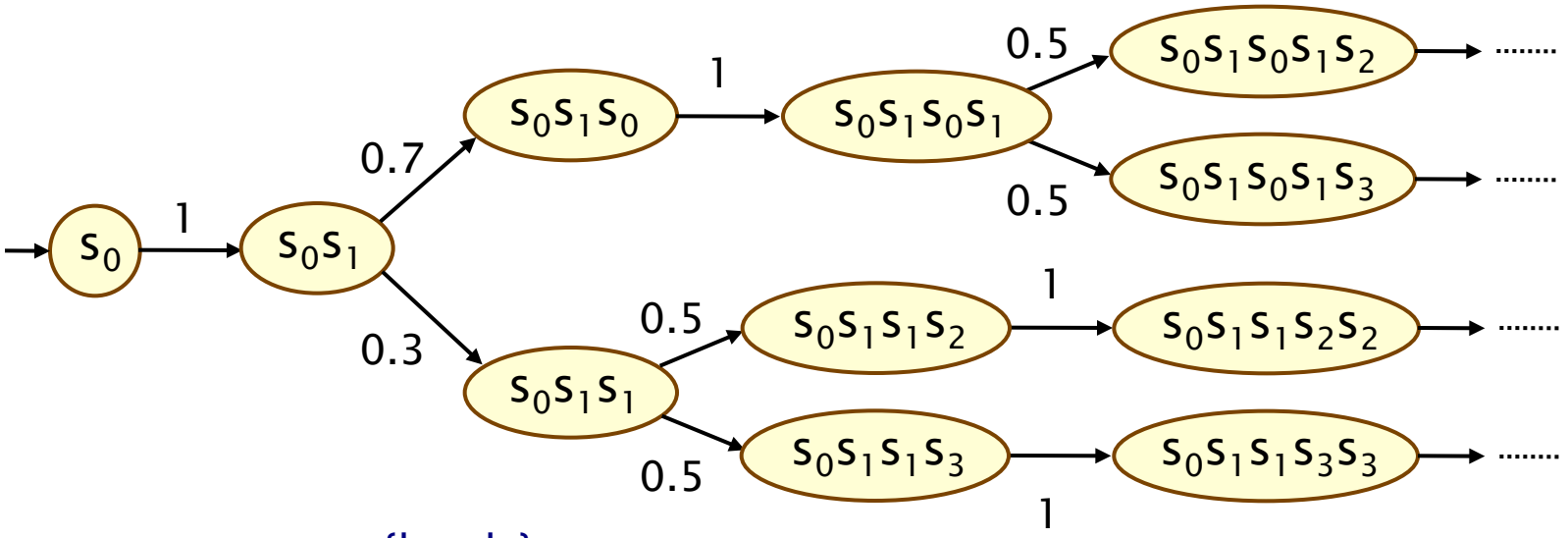
- E.g. strategy that picks b then c in s_1
  - $\sigma(s_0s_1)=(b,\mu_b)$, $\sigma(s_0s_1s_1)=(c,\mu_c)$, $\sigma(s_0s_1s_0s_1)=(c,\mu_c)$

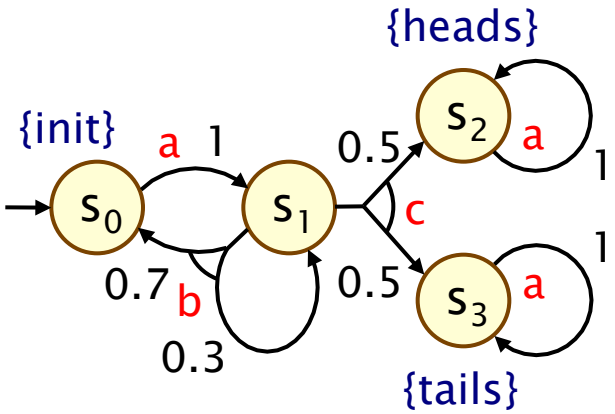
- A strategy results in a fully probabilistic model
  - i.e. an (infinite-state) DTMC over finite paths
  - on which we can define a probability space over infinite paths

- Strategy σ is simple (memoryless) iff $\sigma(s_1…s_n) = \sigma(s_n)$
  - in this case, resulting model reduces to finite Markov chain

Example strategy

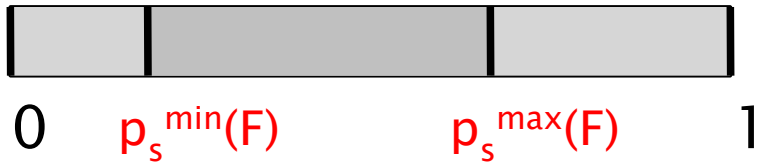
- Fragment of DTMC for strategy which picks b then c in s_1





Probabilistic reachability for MDPs

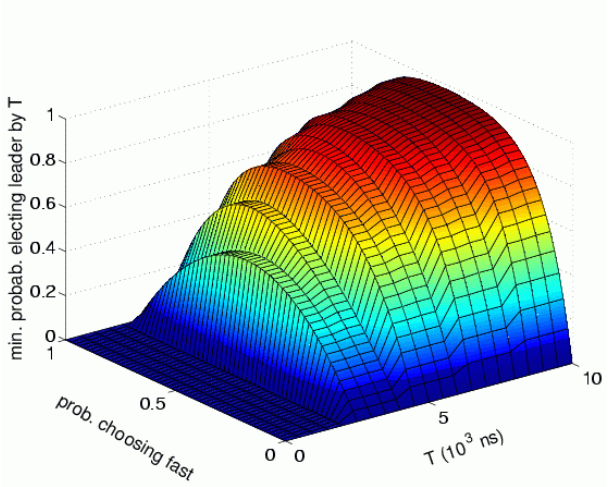
- A strategy σ induces, for each state s in the MDP:
  - a set of infinite paths $Path^{\sigma}(s)$
  - a probability space Pr^{σ}_{s} over $Path^{\sigma}(s)$
- Probabilistic reachability (for a set of goal states $F \subseteq S$)
  - probability of reaching F from state s under strategy σ
  - $p_s^{\sigma}(F) = Pr^{\sigma}_{s} \{ s_0 s_1 s_2 s_3 \ldots \in Path^{\sigma}(s) \mid s_i \in F \text{ for some } i \}$
- Minimum/maximum probabilities over all strategies
  - $p_s^{min}(F) = \inf_{\sigma} p_s^{\sigma}(F)$
  - $p_s^{max}(F) = \sup_{\sigma} p_s^{\sigma}(F)$
  - simple strategies suffice



- Used to reason about best/worst-case behaviour
  - e.g. “maximum probability of an error occurring”

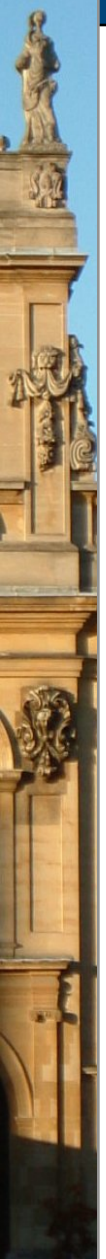
Probabilistic model checking for MDPs

- Also: Bounded reachability properties
  - e.g. “min. probability of algorithm termination within T steps”
- Also: Cost- and reward-based properties
  - augment states/transitions of MDP with real-valued costs
  - define properties as random variables over $Path^{\sigma}(s)$
  - e.g. “max. expected power consumption for the duration of the protocol”
- Probabilistic temporal logics
  - e.g. PCTL extends CTL
- We focus on quantitative analysis
  - i.e. just compute $p_s^{min}(F)$ and $p_s^{max}(F)$
  - useful to spot patterns/trends



Probabilistic model checking for MDPs

- Probabilistic reachability is efficiently computable
  - linear optimisation problem (polynomial complexity)
  - or value iteration (dynamic programming) – simple iterative numerical method; more efficient in practice
  - best/worst case simple strategy can also be generated
- Mature tool support exists, e.g. PRISM
  - efficient (e.g. symbolic) implementations
  - successful application to wide range of application domains
- But major challenges remain, e.g.
  - state-space explosion
  - automating model extraction



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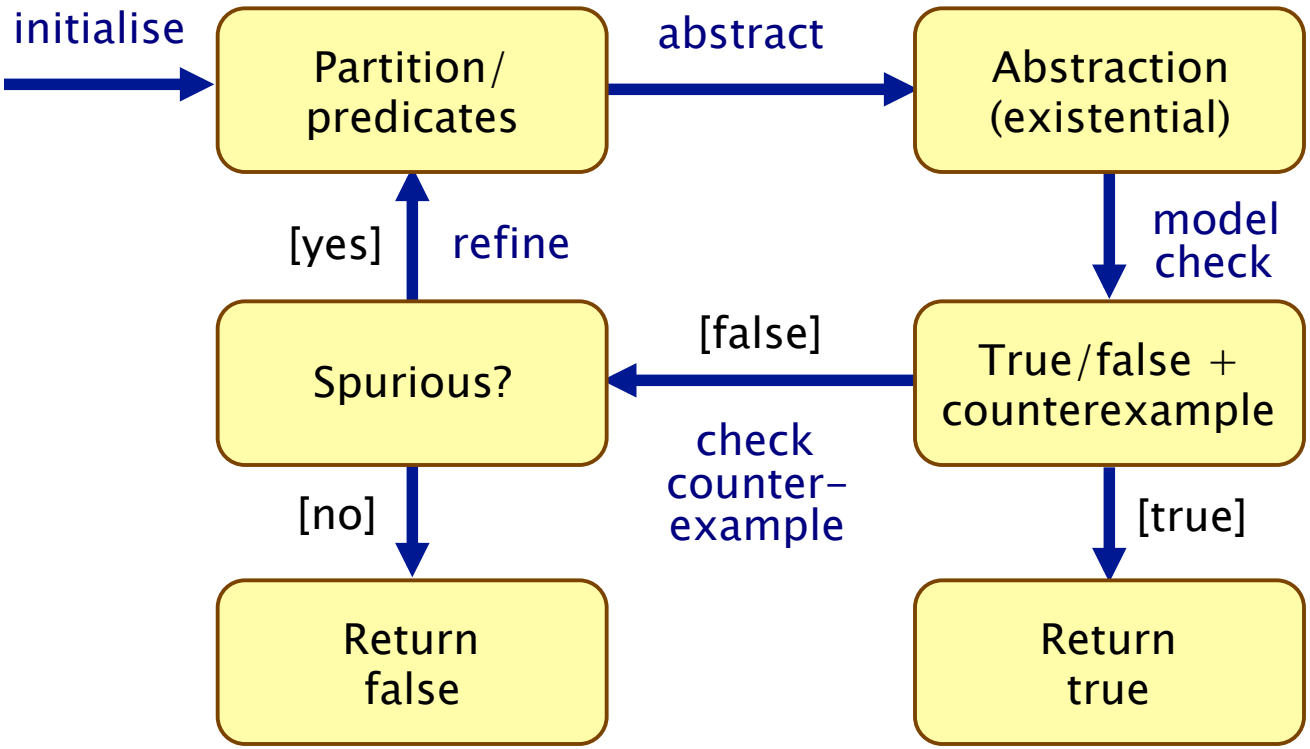
Abstraction

- Very successful in (non-probabilistic) model checking
  - essential for verification of large/infinite-state systems
- Construct abstract model A of concrete model M
  - details not relevant to some property of interest removed
  - e.g. partition of state space based on a set of predicates
- Non-probabilistic case: existential abstraction
  - conservative: existence of path in M implies existence in A
  - hence can model check A to verify safety properties of M
- Abstraction refinement
  - automate process of constructing abstraction
  - start with simple coarse abstraction, then iteratively refine

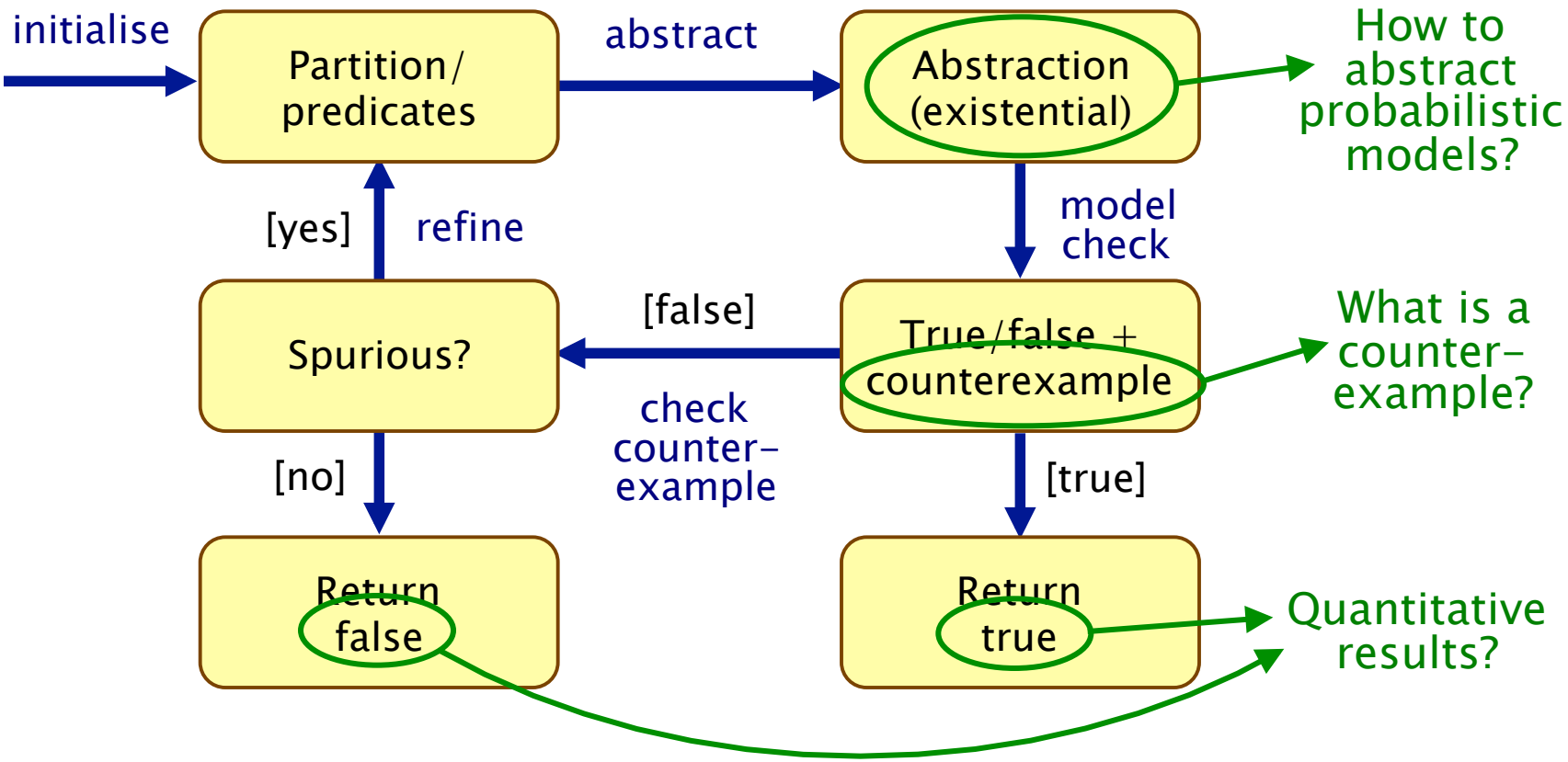
Abstraction + CEGAR

- Counterexample-guided abstraction refinement
  - (non-probabilistic) model checking of reachability properties



Abstraction + CEGAR

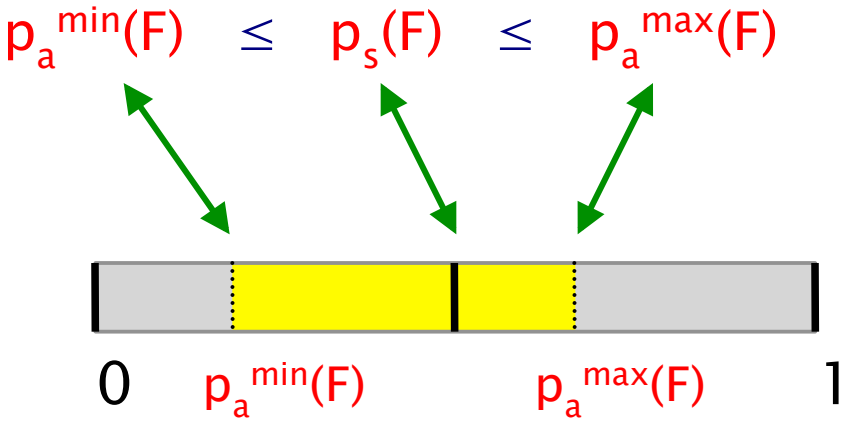
- Counterexample-guided abstraction refinement
  - ~~(non-)~~probabilistic) model checking of reachability properties



Abstraction of DTMCs

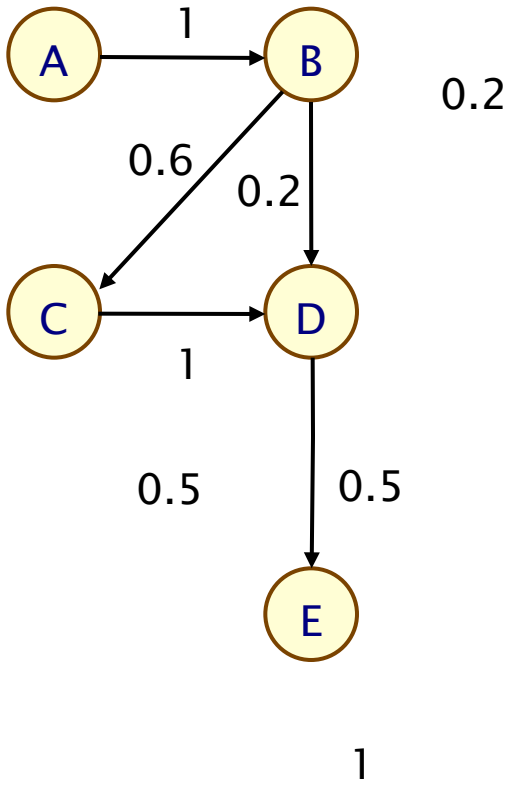
- We use MDPs as abstractions of DTMCs
  - based on a partition A of the (concrete) state space S
  - i.e. each element $a \in A$ is an abstract state

- Analysis of MDP yields lower/upper bounds:
  - for target $F \in A$, $s \in S$ and $a \in A$ with $s \in a$



DTMC → MDP

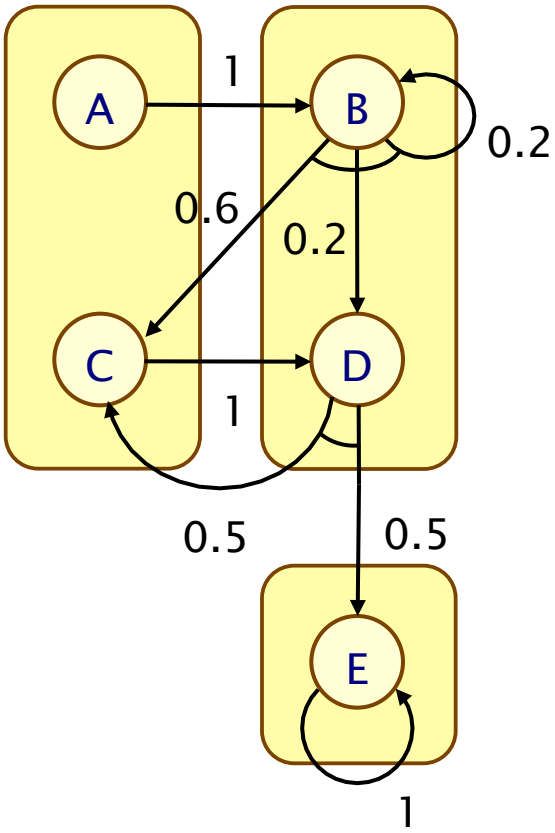
- Partition of (concrete) state space gives abstract states



Concrete model (DTMC)

DTMC → MDP

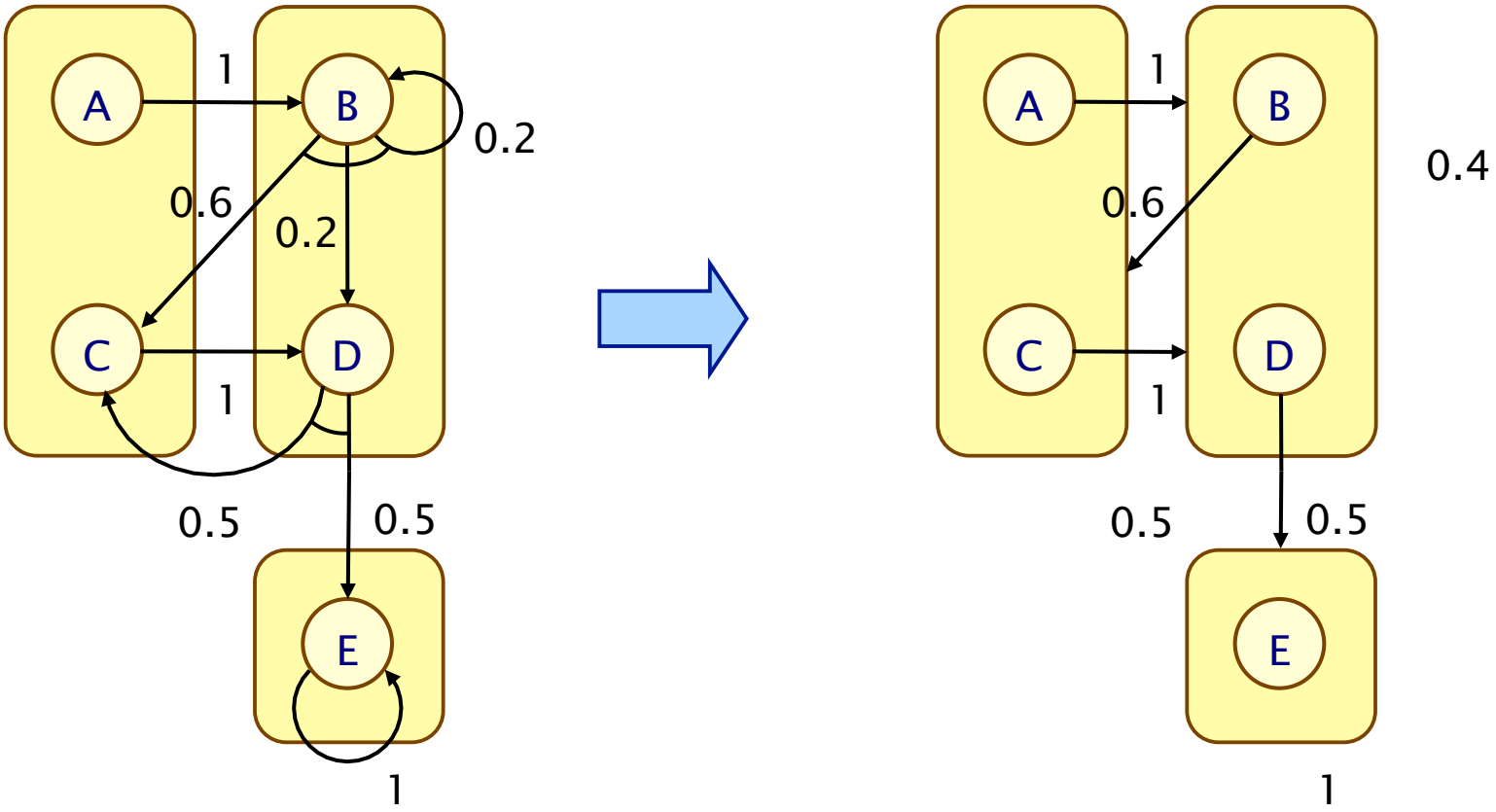
- Partition of (concrete) state space gives abstract states



Concrete model (DTMC)

DTMC → MDP

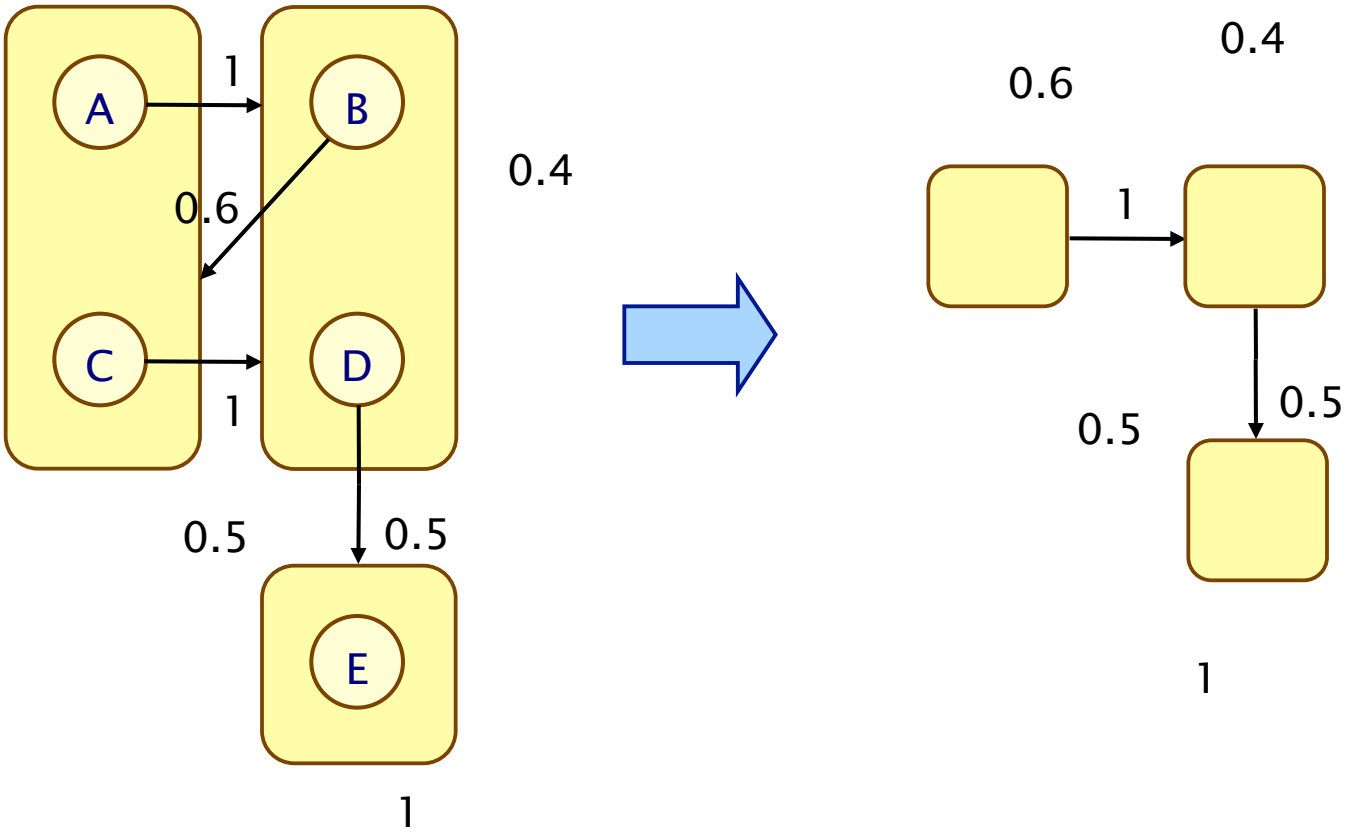
- Distributions are lifted to the abstract state space



Concrete model (DTMC)

DTMC → MDP

- Choices in abstract states become choices in MDP



Concrete model (DTMC)

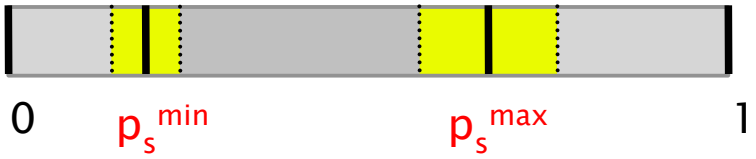
Abstraction (MDP)

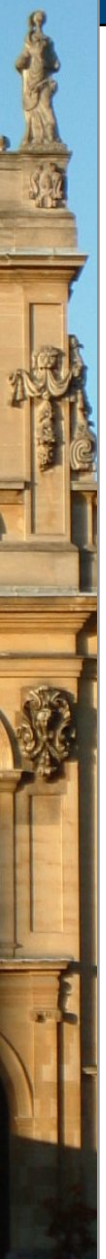
Abstraction of MDPs

- Abstraction increases degree of nondeterminism
  - i.e. minimum probabilities are lower and maximums higher

  - what form does the abstraction of an MDP take?

- Our approach: two-player stochastic games [QEST'06]
- Key idea: separate two forms of nondeterminism
  - (a) from abstraction and (b) from original MDP
  - then generate separate lower/upper bounds for min/max



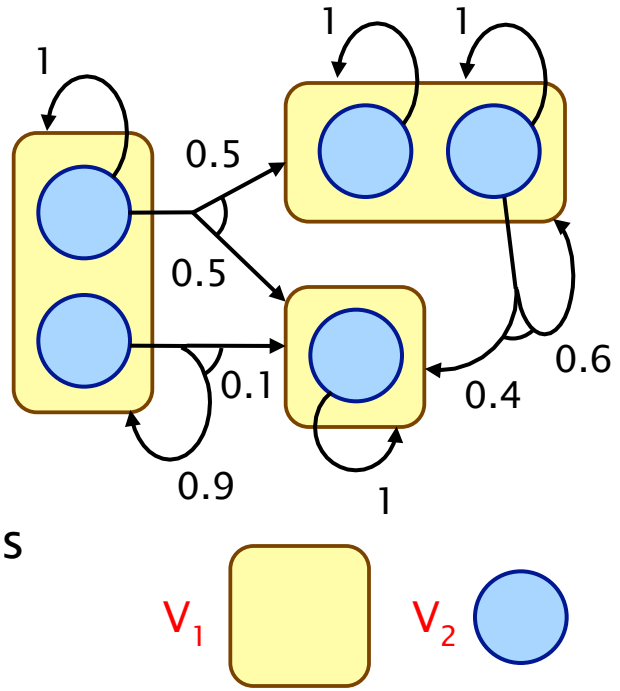


Two-player stochastic games

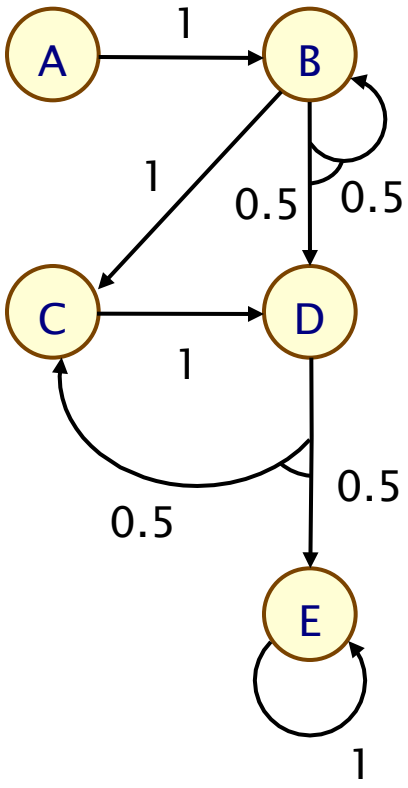
- Subclass of simple stochastic games [Shapley], [Condon]
  - two nondeterministic players and probabilistic choice
  - vertices V of game partitioned into V_1 and V_2 (player 1/2 vertices)
  - player 1 has choices between V_2 vertices
  - player 2 has choices between distributions over V_1 vertices
- A resolution of the nondeterminism in the game
  - corresponds to a pair of strategies for players 1 and 2: (σ_1,σ_2)
  - $p_v^{\sigma 1,\sigma 2}(F)$ = probability of reaching F from v under (σ_1,σ_2)
- Optimal reachability probabilities for player 1 and player 2
  - informally: the maximum probability of reaching F a player can guarantee, no matter what the other player does
  - formally: $\sup_{\sigma 1} \inf_{\sigma 2} p_v^{\sigma 1,\sigma 2}(F)$ and $\sup_{\sigma 2} \inf_{\sigma 1} p_v^{\sigma 1,\sigma 2}(F)$
  - computable (and simple strategies) with value iteration

Games as abstractions of MDPs

- Abstraction of an MDP is a two-player stochastic game
  - based on a partition A of the concrete state space
- where:
  - player 1 controls the nondeterminism of the abstraction
  - player 2 controls the nondeterminism of the MDP
- Player 1 vertices: abstract states
  - elements of partition A
- Player 2 vertices: choices of MDP lifted to abstract states
  - sets of distributions over A
  - or equivalently, sets of concrete states with the same abstract choices

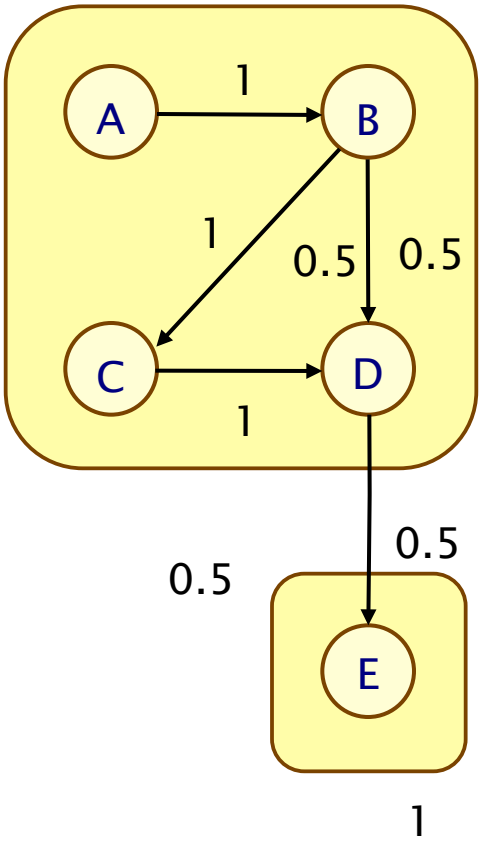


MDP → Game



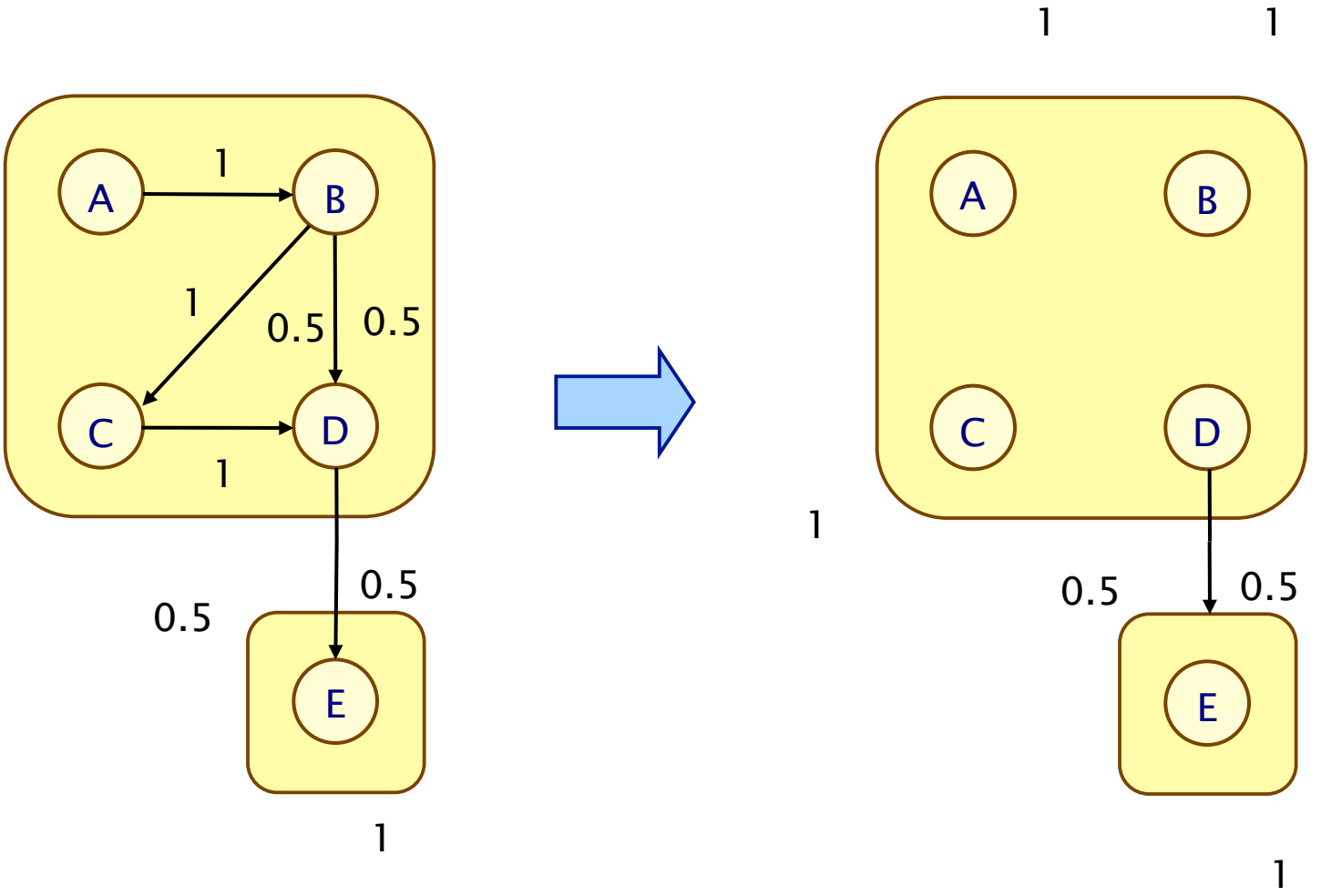
MDP → Game

- Player 1 vertices are partition elements (abstract states)



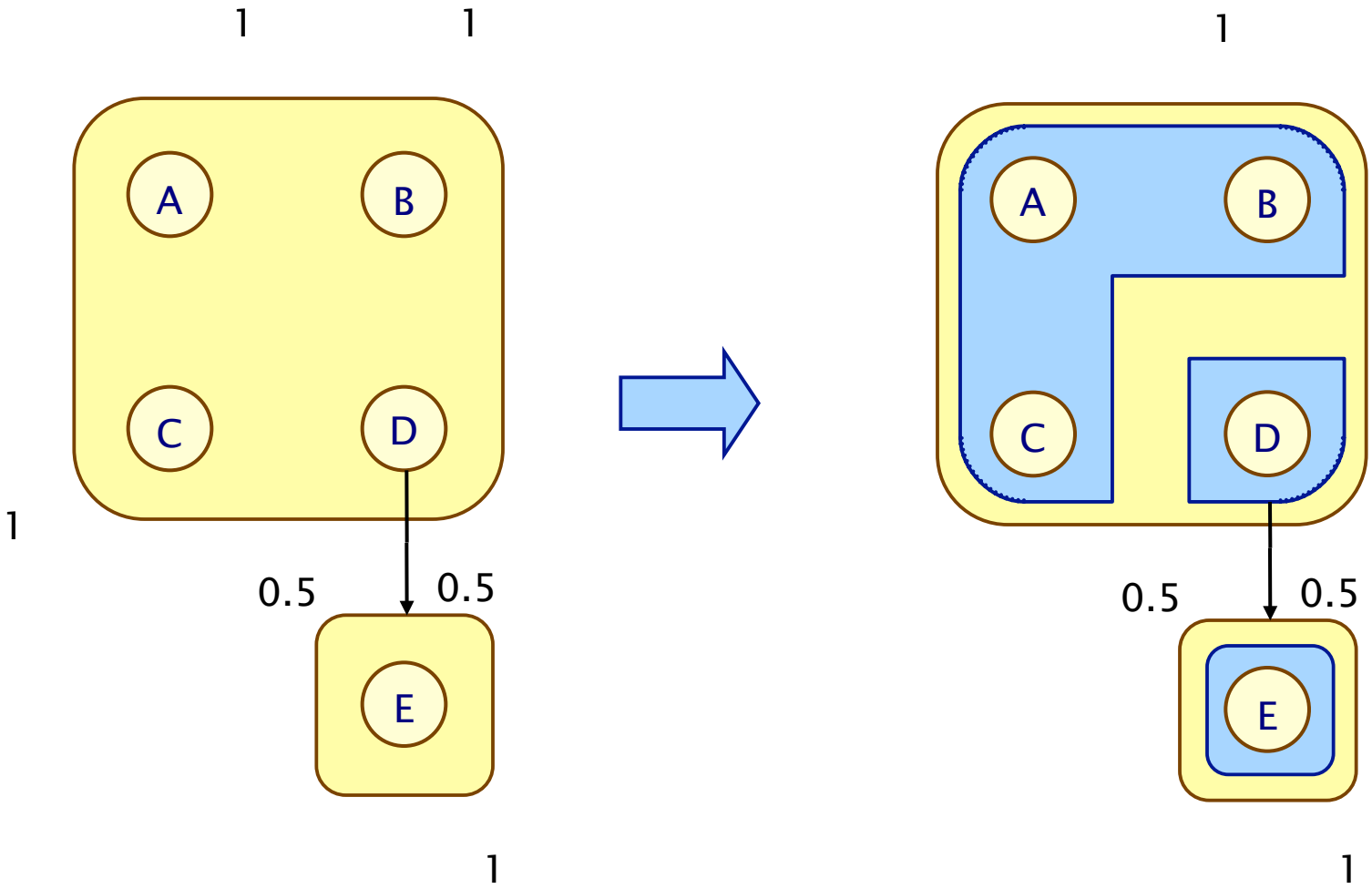
MDP → Game

- (Sets of) distributions are lifted to the abstract state space



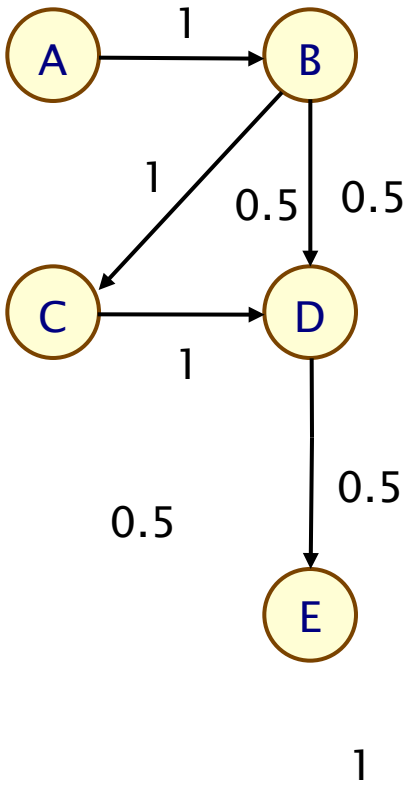
MDP → Game

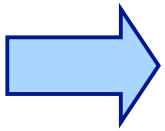
- States with same (sets of) choices form player 2 vertices

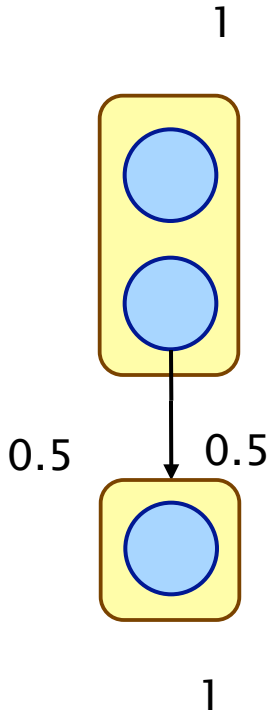


MDP → Game

- Complete transformation:







Concrete model (MDP)

Abstraction (game)

Analysis of the abstraction

- Analysis of game yields lower/upper bounds:
  - for target $F \in A$, $s \in S$ and $a \in A$ with $s \in a$

$$\inf_{\sigma 1,\sigma 2} p_v^{\sigma 1,\sigma 2}(F) \leq p_s^{min}(F) \leq \sup_{\sigma 1} \inf_{\sigma 2} p_v^{\sigma 1,\sigma 2}(F)$$

$$\sup_{\sigma 2} \inf_{\sigma 1} p_v^{\sigma 1,\sigma 2}(F) \leq p_s^{max}(F) \leq \sup_{\sigma 1,\sigma 2} p_v^{\sigma 1,\sigma 2}(F)$$

Analysis of the abstraction

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min/max reachability probabilities for original MDP

Analysis of the abstraction

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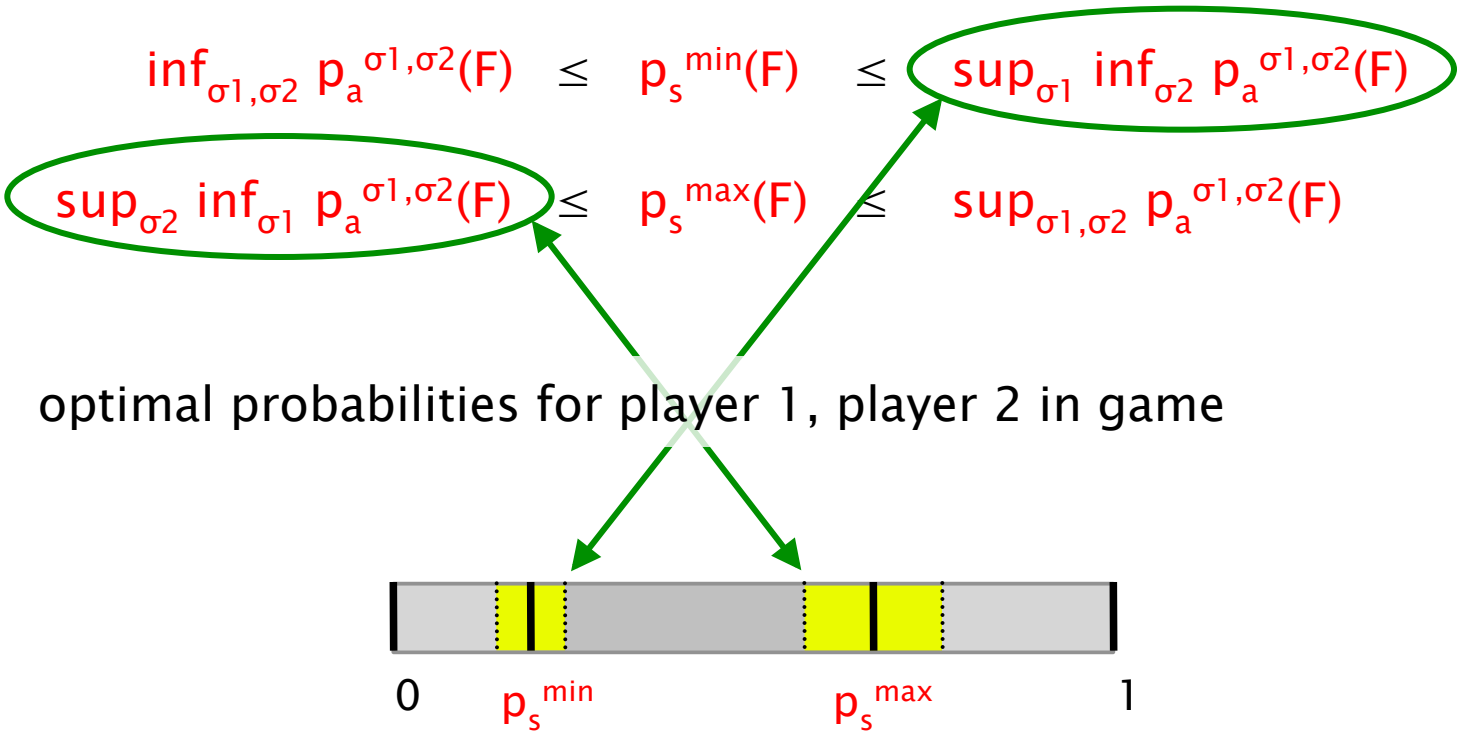
  - for target $F \in A$, $s \in S$ and $a \in A$ with $s \in a$



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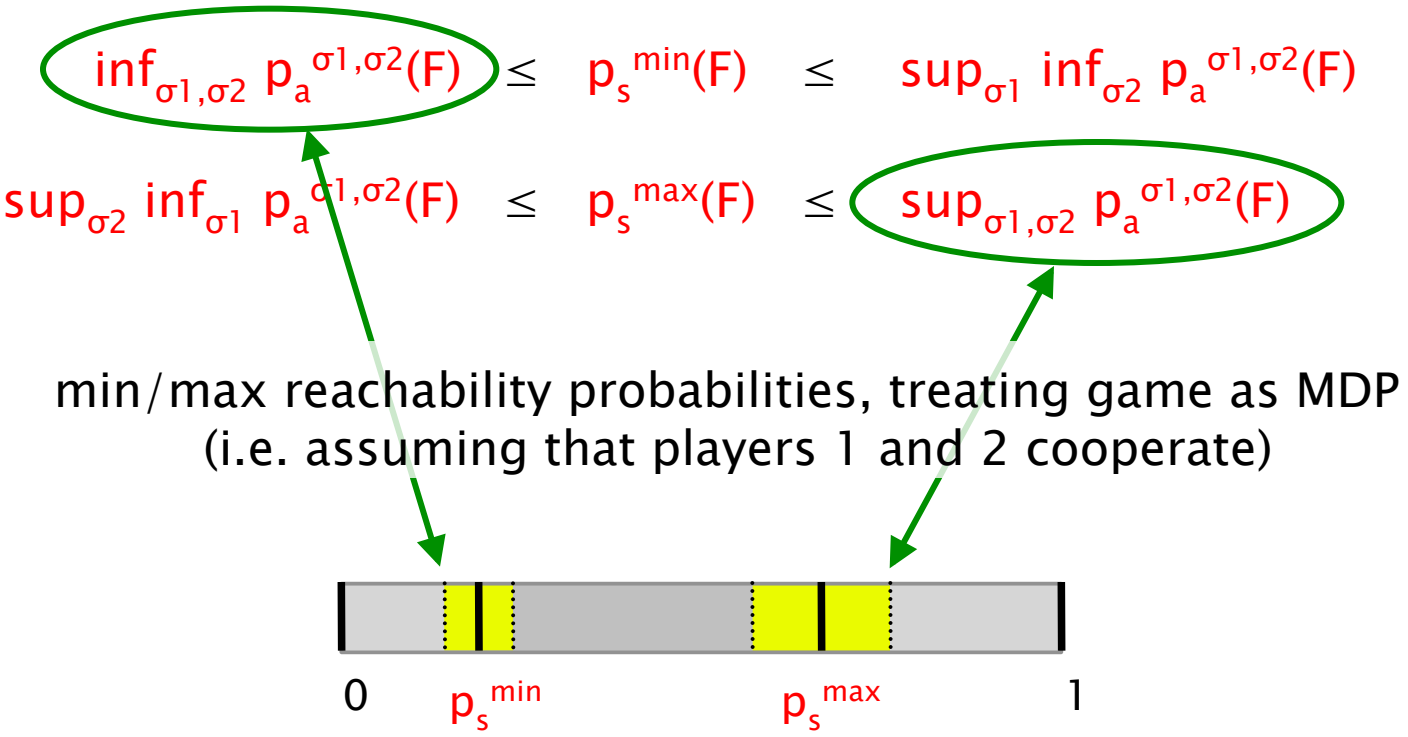
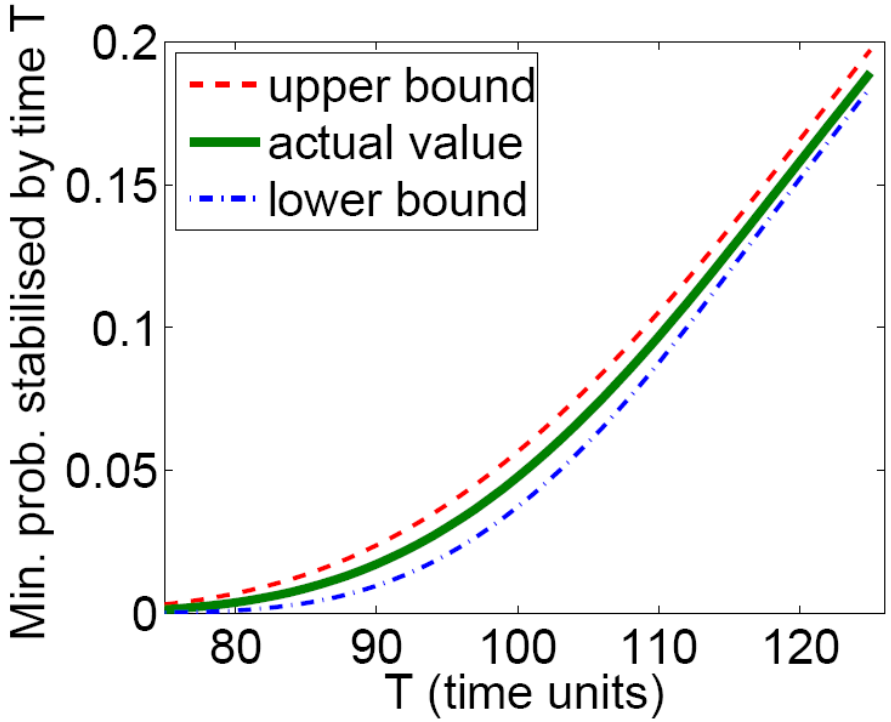


Abstraction: Results

- Israeli & Jalfon's Self Stabilisation [IJ90]
  - protocol for obtaining a stable state in a token ring
  - minimum probability of reaching a stable state by time T

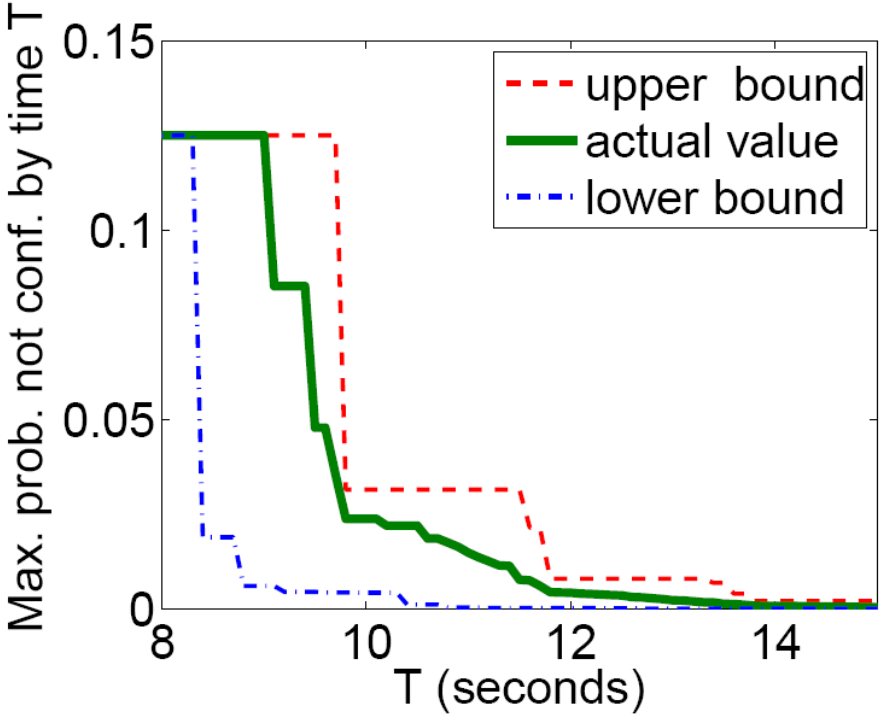


concrete states: 1,048,575

abstract states: 627

Abstraction: Results

- IPv4 Zeroconf [CAG02]
  - protocol for obtaining an IP address for a new host
  - maximum probability the new host not configured by T

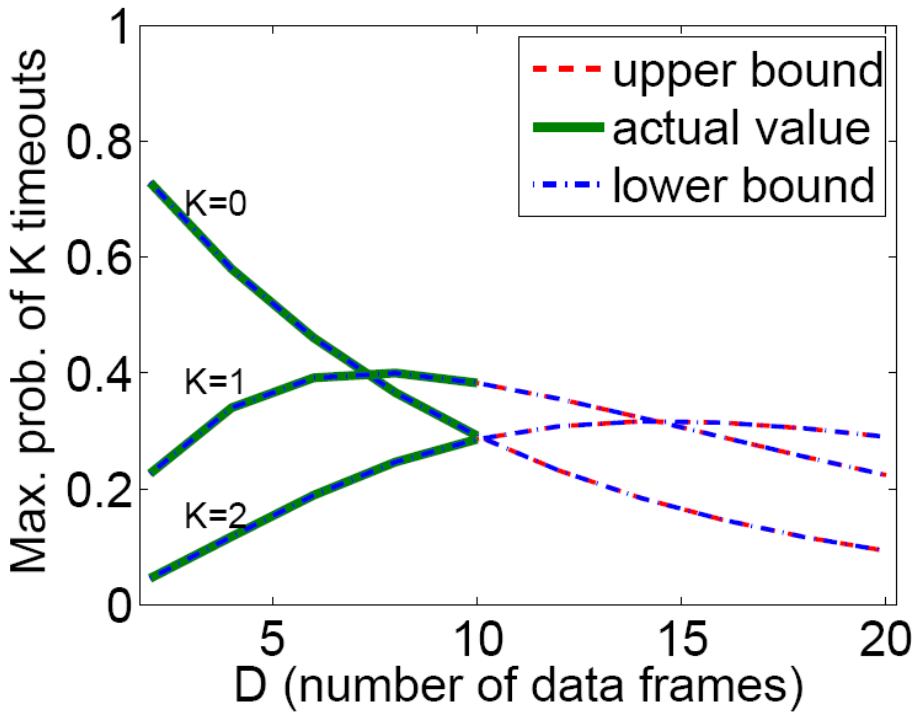


concrete states: 838,905

abstract states: 881

Abstraction: Results

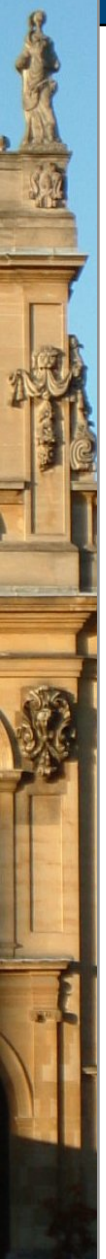
- Sliding Window Protocol
  - protocol for sending data over an insecure medium
  - maximum probability of K timeouts



D=8
concrete states: 189,952
abstract states: 742

D=10
concrete states: 987,136
abstract states: 964

D=20
concrete states: ??
abstract states: 2,074



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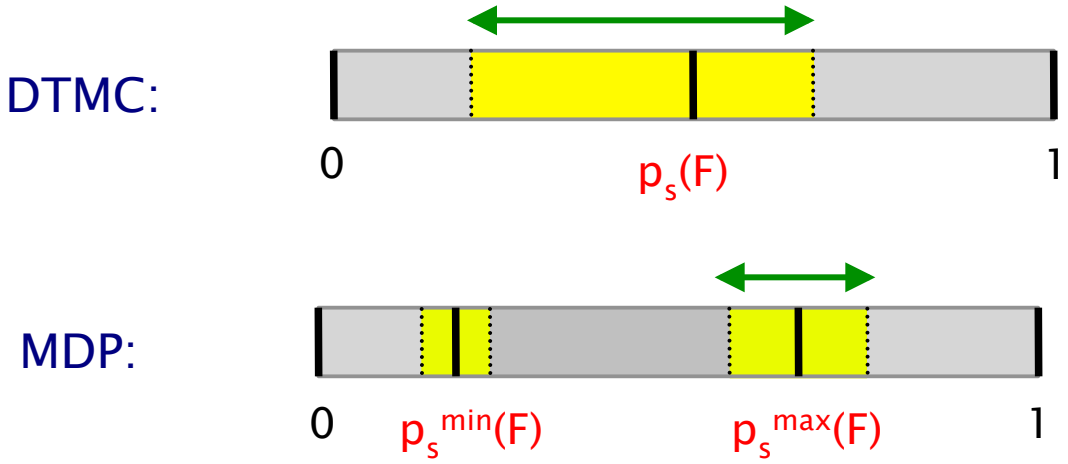
  - abstractions of MDPs (two-player stochastic games)
- Quantitative abstraction refinement
  - abstraction-refinement loop
  - verification of probabilistic software, real-time systems
- Current/future work & Conclusions

Abstraction refinement

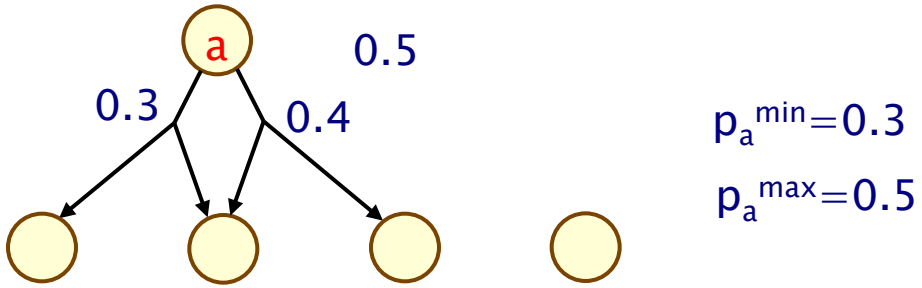
- Consider (max) difference between lower/upper bounds
  - gives a quantitative measure of the abstraction's precision



- If the difference ("error") is too great, refine the abstraction
  - a finer partition yields a more precise abstraction
  - model checking results (bounds/strategies) guide refinement

Abstraction refinement

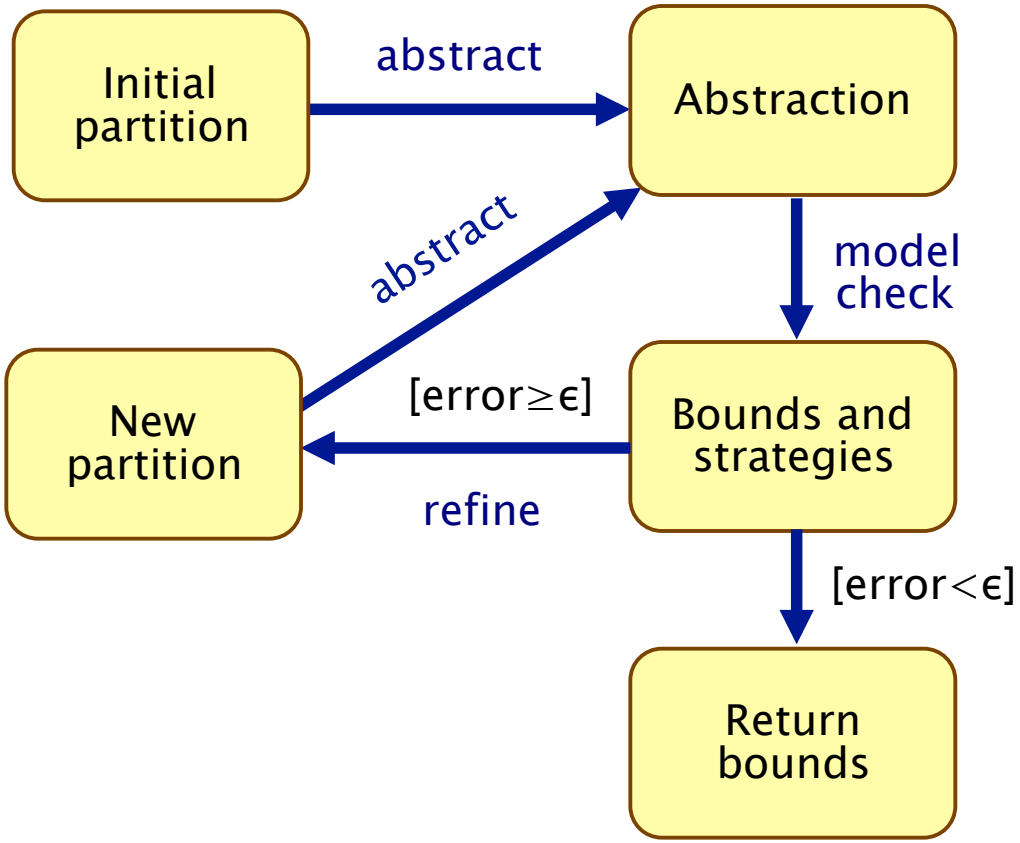
- Probabilistic reachability yields simple strategies
- Consider abstract state a with "error" 0.2:



- Abstract state a represents a set of concrete states
  - each choice in a corresponds to a subset of these
- Refine (split) a accordingly:
  - "strategy-based" refinement or "value-based" refinement

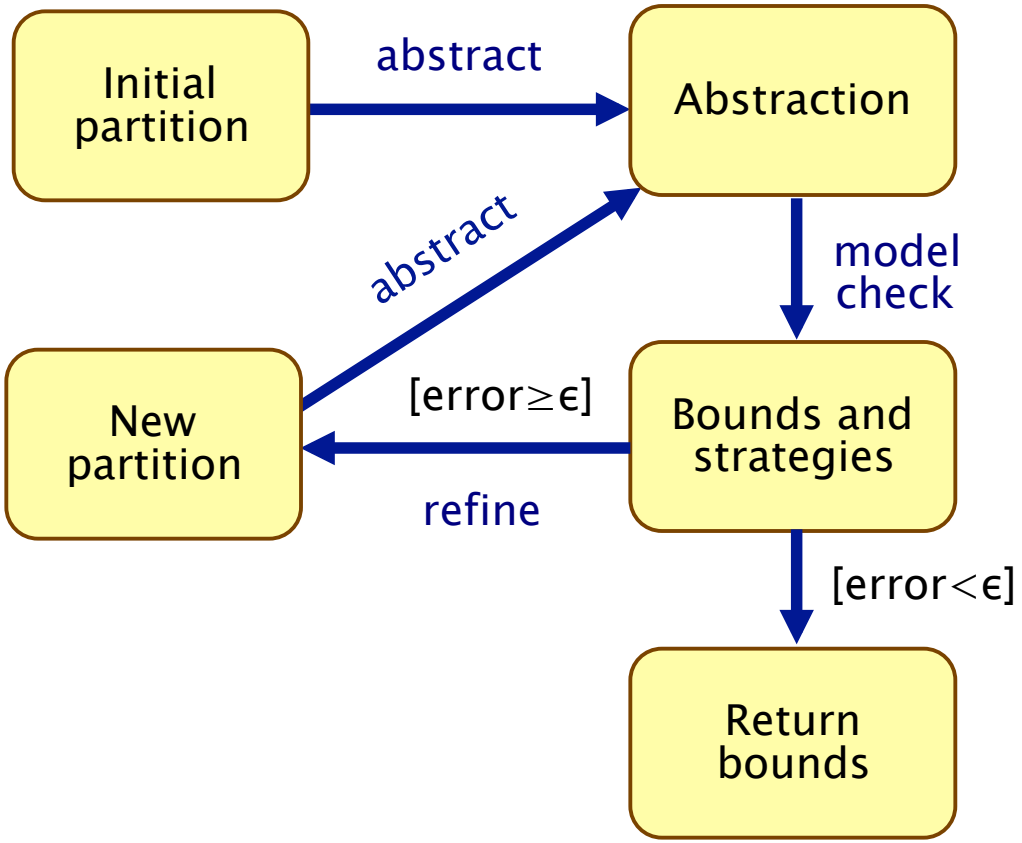
Abstraction-refinement loop

- Quantitative abstraction-refinement loop for MDPs



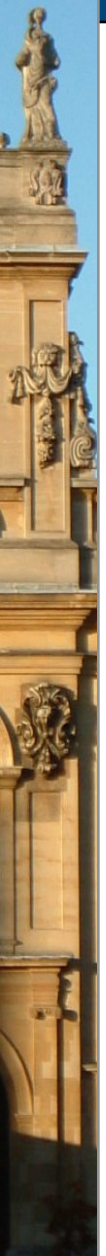
Abstraction-refinement loop

- Quantitative abstraction-refinement loop for MDPs



- Refinements yield strictly finer partition
- Guaranteed to converge for finite models
- Guaranteed to converge for infinite models with finite bisimulation

Abstraction-refinement loop



- Implementations of quantitative abstraction refinement…
- 1. Explicit-state prototype implementation
  - confirms viability of game-based abstraction/refinement
- 2. Compositional abstraction of PRISM models [QAPL'08]
  - efficient construction of (manually specified) abstractions
- 3. Verification of probabilistic software [VMCAI'09]
  - (predicate) abstraction/refinement of infinite-state MDPs
- 4. Verification of probabilistic timed automata [FORMATS'09]
  - (DBM-based) abstraction/refinement of infinite-state MDPs

Probabilistic software

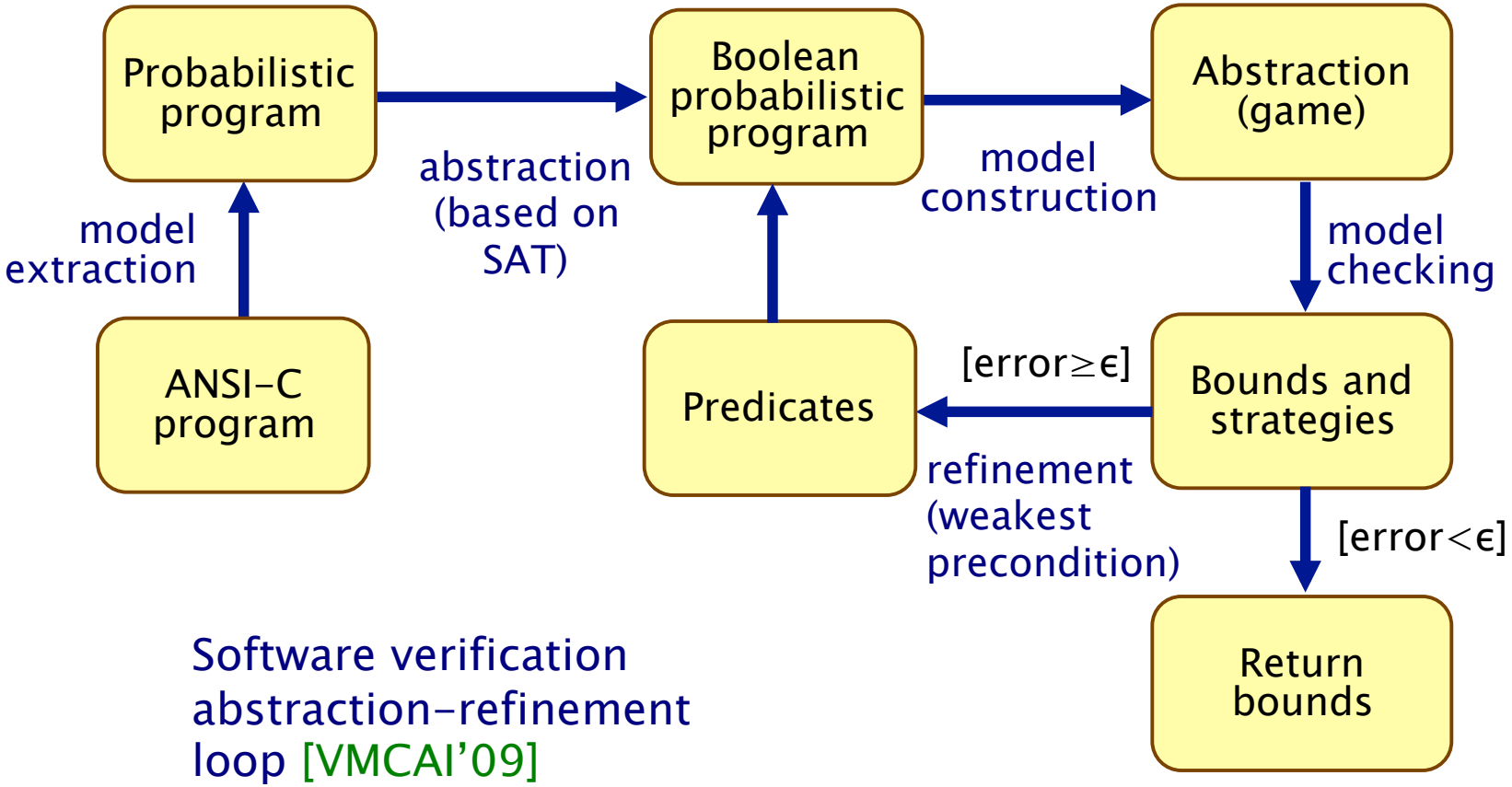
- Consider sequential ANSI C programs
  - support functions, pointers, arrays, but not dynamic memory allocation, unbounded recursion, floating point operations
  - probabilistic functions (for failures, randomisation)
  - nondeterministic functions (e.g. to abstract system calls)
  - infinite-state MDP semantics
- Quantitative properties based on probabilistic reachability
  - e.g. maximum probability of unsuccessful data transmission
  - e.g. minimum expected number of packets sent
- Prototype tool qprover
  - components from PRISM (model checking of stochastic games)
  - components from SATABS (predicate abstraction, SAT solvers)
  - analysed Linux network utilities (ping, tftp) – approx 1KLOC

Example – sample target program

```
bool fail = false;
int c = 0;
int main ()
{
      // nondeterministic
      c = num_to_send ();
      while (! fail && c > 0)
      {
            // probabilistic
            fail = send_msg ();
            c --;
      }
}
```

Φ: “what is the minimum/maximum probability of the program terminating with fail being true?”

Abstraction-refinement loop

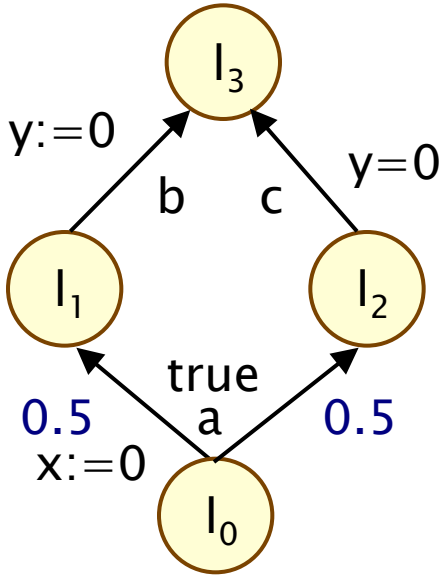


Software verification abstraction-refinement loop [VMCAI'09]

Verification of PTAs

- Verification of probabilistic timed automata (PTAs)
  - discrete states, discrete probabilistic choice, nondeterminism and real-valued clocks
  - infinite-state MDP semantics

- Timed automata + probabilities

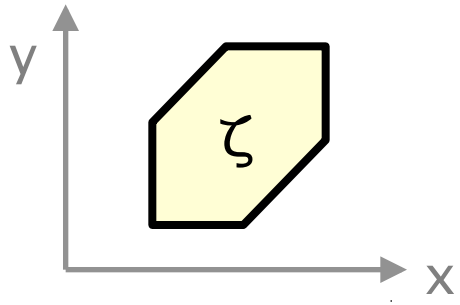


- PRISM modelling language + clocks

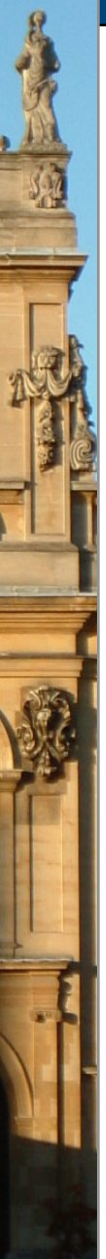
```
module M
    s : [0..3];
    x : clock;
    [a] s=0 & x<10 → (s'=1);
    [b] s=1 → 0.5:(s'=2) + 0.5:(s'=3)&(x'=0);
endmodule
```

Verification of PTAs

- Finite-state (game) abstractions using zones
  - efficiently represented as difference bound matrices (DBMs)



- Initial abstraction from forwards reachability
  - subsequent refinements through zone splitting
- Properties:
  - guaranteed to converge in finite time for any $\epsilon \geq 0$
  - i.e. exact verification of PTAs
- Outperforms existing PTA verification techniques
  - on several large case studies



Overview

- Probabilistic model checking
  - discrete-time Markov chains (DTMCs)
  - Markov decision processes (MDPs)
  - probabilistic reachability
- Abstraction for probabilistic models
  - abstractions of DTMCs (MDPs)
  - abstractions of MDPs (two-player stochastic games)
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Nondeterministic abstractions

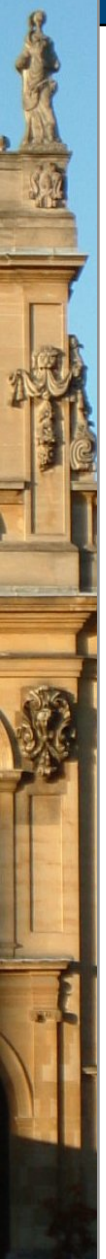
- We can consider a general class of "nondeterministic" abstractions for probabilistic models

Concrete model:		Abstraction:
DTMC	→	MDP
MDP	→	TPSG
CTMC	→	CTMDP
CTMDP	→	CTTPSG

- CTMDP = continuous-time Markov decision process
- CTTPSG = continuous-time two-player stochastic game

Abstraction for CTMCs

- A useful subset of CTMC properties are untimed
  - unbounded probabilistic reachability
  - steady-state (long-run) probabilities
  - so can abstract embedded DTMC as an MDP
- More interesting: time-bounded (transient properties)
  - can abstract CTMCs as CTMDPs, yielding lower/upper bounds
  - [Smith], [Katoen et al.], …
  - moreover, abstractions are uniform CTMDPs
- But: How to refine?
  - time-bounded reachability does not yield simple strategies
- How to build abstraction? What high-level model?



Conclusions

- Abstraction approach for probabilistic models
  - DTMCs abstracted as MDPs
  - MDPs abstracted as two-player stochastic games
  - abstraction yields lower/upper bounds on probabilities
- Quantitative abstraction refinement
  - bounds give quantitative measure of utility of abstraction
  - bounds/strategies can be used to guide refinement
  - quantitative abstraction-refinement loop (for error $< \epsilon$)
  - fully automatic generation of abstraction
  - works in practice: probabilistic software & timed automata
- Current & future work
  - CTMCs, timed properties
  - probabilistic/stochastic hybrid systems
  - improved refinement heuristics, imprecise abstractions