# Probabilistic Model Checking

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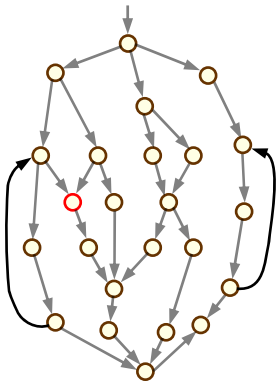
AVOCS '07

# Overview

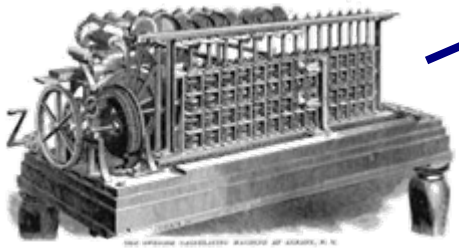
- Introduction and motivation
- Probabilistic models
  - discrete-time Markov chains (DTMCs)
  - Markov decision processes (MDPs)
- Property specification
  - the logic PCTL, quantitative specifications, costs and rewards
- Probabilistic model checking
  - algorithms, implementation, tool support
- Case studies
  - probabilistic contract signing

# Verification via model checking

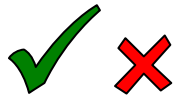
Finite-state model

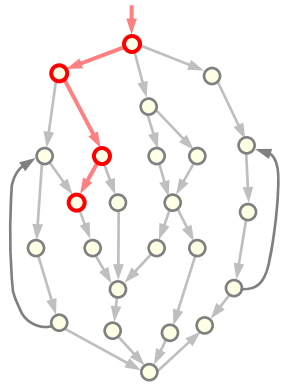


Model checker



Result





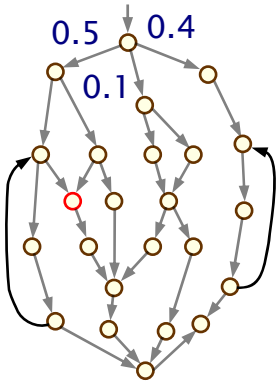
Error trace

¬EF error

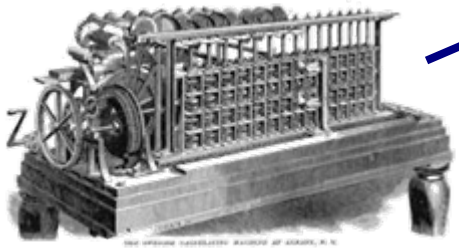
Temporal logic specification

# Probabilistic model checking

Finite-state probabilistic model
e.g. Markov chain




Probabilistic model checker
e.g. PRISM



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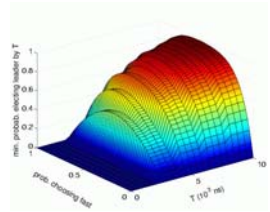

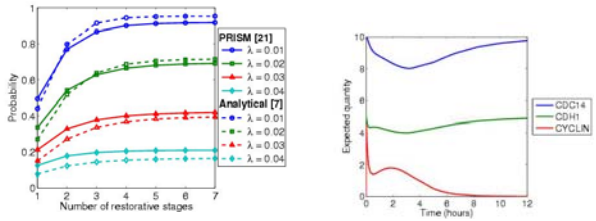


Quantitative results

$P_{<0.01}$ [ F error ]

Probabilistic temporal logic specification
e.g. PCTL

# Why probability?

- Randomisation used in distributed coordination algorithms
  - as a symmetry breaker, e.g. Bluetooth, FireWire, …
  - in gossip routing to reduce flooding
- To model uncertainty and performance
  - failure rates, e.g. communication protocols
  - component life-times, e.g. network cluster
  - inter-arrival times, e.g. queuing systems
  - biochemical reaction rates
- Requirements are quantitative as well as qualitative
  - Quality of Service: how reliable is my car's Bluetooth network?
  - how efficient is my phone's power management policy?
  - quantification of trust, anonymity, …

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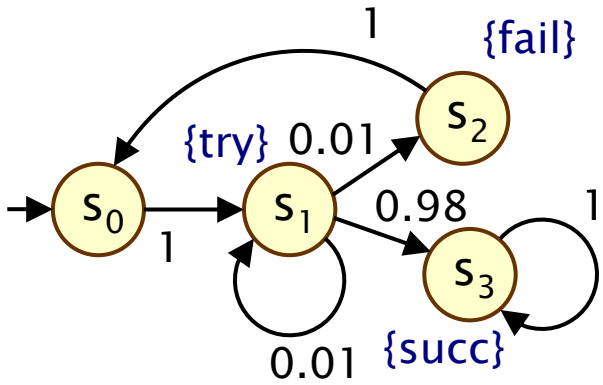
# Discrete-time Markov chains (DTMCs)

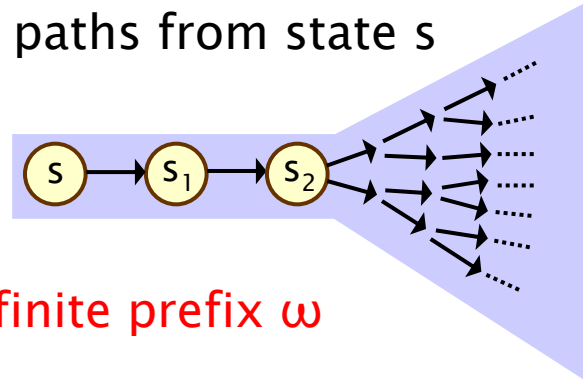
- State-transition systems augmented with probabilities
  - discrete set of states representing possible system configurations
  - transitions between states occur in discrete time-steps
  - probability of making transitions between states is given by discrete probability distributions




- Formally, a DTMC D is a tuple $(S, s_{init}, \mathbf{P}, L)$
  - $S$ is a finite set of states, $s_{init} \in S$ is the initial state
  - $\mathbf{P} : S \times S \rightarrow [0,1]$ is the transition probability matrix where $\Sigma_{s' \in S} \mathbf{P}(s,s') = 1$ for all $s \in S$
  - $L : S \rightarrow 2^{AP}$ labels states with atomic propositions

# Paths and probabilities

- A (finite or infinite) path through a DTMC
  - is a sequence of states $s_0 s_1 s_2 s_3 \ldots$ such that $\mathbf{P}(s_i, s_{i+1}) > 0 \ \forall i$
  - represents an execution (i.e. one possible behaviour) of the system which the DTMC is modelling
- To reason (quantitatively) about this system
  - need to define a probability space over paths [KSK76]
- Intuitively:
  - sample space: Path(s) = set of all infinite paths from state s
  - events: sets of infinite paths from s
  - basic events: cylinder sets (or "cones")
  - cylinder set $C(\omega)$, for a finite path $\omega$ = set of infinite paths with the common finite prefix $\omega$
  - probability measure $Pr_s$ over Path(s)
  - for example: $Pr_s(C(ss_1s_2) = \mathbf{P}(s,s_1) \cdot \mathbf{P}(s_1,s_2)$

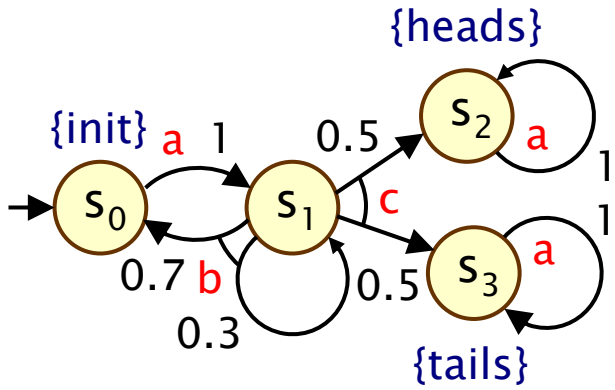


# Adding nondeterminism

- But, some aspects of a system may not be probabilistic and should not be modelled probabilistically; for example:
- Concurrency – scheduling of parallel components
  - e.g. randomised distributed algorithms – multiple probabilistic processes operating asynchronously
- Unknown environments
  - e.g. probabilistic security protocols – unknown adversary
- Underspecification – unknown model parameters
  - e.g. a probabilistic communication protocol designed for message propagation delays of between $d_{min}$ and $d_{max}$

# Markov decision processes (MDPs)

- Extension of DTMCs which allow nondeterministic choice
  - discrete set of states
  - transitions are discrete time-steps
  - nondeterministic choice between several discrete probability distributions over successor states



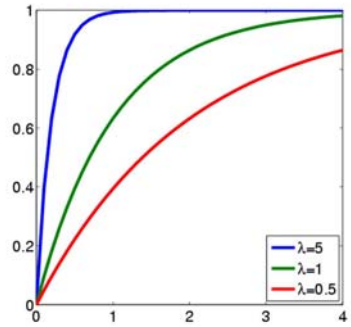

- Formally, an MDP M is a tuple $(S, s_{init}, \mathbf{Steps}, L)$ where:
  - States S, initial state $s_{init}$ and labelling function L as for DTMCs
  - **Steps** : $S \rightarrow 2^{Act \times Dist(S)}$ is the transition probability function

    where Act is a set of actions and Dist(S) is the set of discrete probability distributions over the set of states S

# Paths, probabilities, adversaries

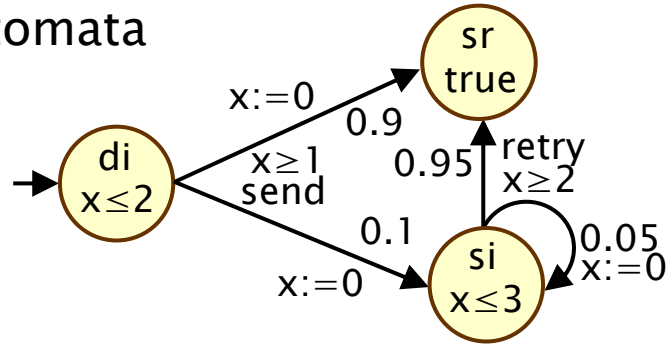
- A (finite or infinite) path through an MDP
  - is a sequence of states and action/distribution pairs
  - $s_0(a_0,\mu_0)s_1(a_1,\mu_1)s_2\ldots$ where $(a_i,\mu_i) \in \mathbf{Steps}(s_i)$ and $\mu_i(s_{i+1}) > 0$
  - note both nondeterministic and probabilistic choice resolved
- To reason about probabilistic behaviour in an MDP
  - first resolve nondeterminism with an adversary
  - adversaries are also known as "schedulers" or "policies"
  - formally, an adversary is a mapping from any finite path in the MDP to a subsequent action/distribution pair
- For an adversary A
  - the MDP reduces to a (possibly infinite state) DTMC
  - the resulting set of infinite paths is $\mathrm{Path}^A(s)$
  - over which we can define a probability measure $\Pr^A_s$

# Other models




- Continuous-time Markov chains (CTMCs)
  - discrete state space, continuous time
  - exponentially distributed delays
  - suited to modelling component lifetimes, inter-arrival times, biochemical reaction rates, …

- Probabilistic timed automata (PTAs)
  - probabilistic extension of timed automata
  - discrete states, real-time clocks, discrete probability distributions, nondeterminism




- Others: continuous-time MDPs, interactive Markov chains, …

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# PCTL

- Temporal logic for describing properties of DTMCs/MDPs
  - PCTL = Probabilistic Computation Tree Logic [HJ94]
  - essentially the same as the logic pCTL of [ASB+95]
- Extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators
- Example
  - send → $P_{\geq 0.95}$ [ $F^{\leq 10}$ deliver ]
  - "if a message is sent, then the probability of it being delivered within 10 steps is at least 0.95"

# PCTL syntax

- $\phi$ ::= true | a | $\phi \wedge \phi$ | $\neg\phi$ | $P_{\sim p}$ [ $\psi$ ] (state formulas)
- $\psi$ ::= $X\,\phi$ | $\phi\ U^{\leq k}\ \phi$ | $\phi\ U\ \phi$ (path formulas)

"next"

"bounded until"

"until"

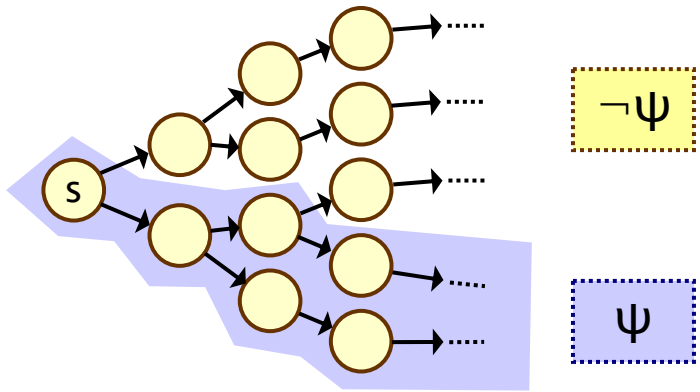
$\psi$ is true with probability ~p

  - where a is an atomic proposition, used to identify states of interest, $p \in [0,1]$ is a probability, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- Usual equivalences:
  - false $\equiv \neg$true, $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$, $\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$
  - Reachability ("eventually"): $F\ \phi \equiv$ true $U\ \phi$, $F^{\leq k}\ \phi \equiv$ true $U^{\leq k}$
  - Invariance ("always"): $G\ \phi \equiv \neg(F\ \neg\phi) \equiv \neg$(true $U\ \neg\phi$)

# PCTL semantics for DTMCs

- PCTL formulas interpreted over states of a DTMC
  - usual (CTL) semantics for most operators
- Semantics of the probabilistic operator P
  - informal definition: $s \models P_{\sim p}[\psi]$ means that "the probability, from state s, that $\psi$ is true for an outgoing path satisfies $\sim p$"
  - example: $s \models P_{<0.25}[X \text{ fail}] \Leftrightarrow$ "the probability of atomic proposition fail being true in the next state of outgoing paths from s is less than 0.25"
  - formally: $s \models P_{\sim p}[\psi] \Leftrightarrow Pr_s\{\omega \in Path(s) \mid \omega \models \psi\} \sim p$



# PCTL semantics for MDPs

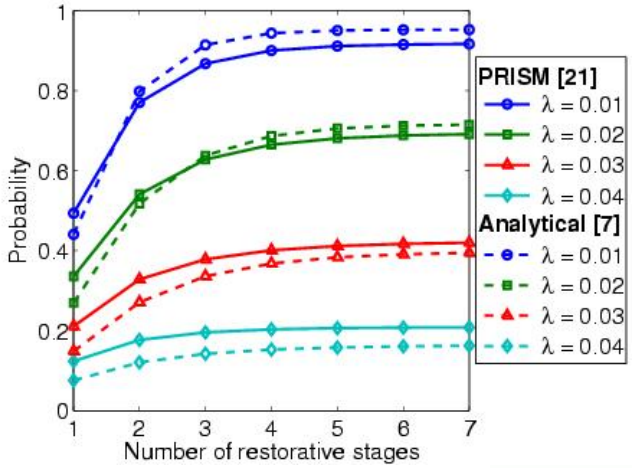
- Semantics of the probabilistic operator P
  - can only define probabilities for a specific adversary A
  - $s \models P_{\sim p}[\psi]$ means "the probability, from state s, that $\psi$ is true for an outgoing path satisfies $\sim p$ for all adversaries"
  - formally $s \models P_{\sim p}[\psi] \Leftrightarrow Prob^A(s, \psi) \sim p$ for all adversaries A
  - where $Prob^A(s, \psi) = Pr^A_s \{ \omega \in Path^A(s) \mid \omega \models \psi \}$
- In practice, we compute:
  - $p_{max}(s, \psi) = \sup_A Prob^A(s, \psi)$
  - $p_{min}(s, \psi) = \inf_A Prob^A(s, \psi)$
- Can also define semantics for a specific class of adversaries
  - e.g. for all fair adversaries

# Quantitative properties

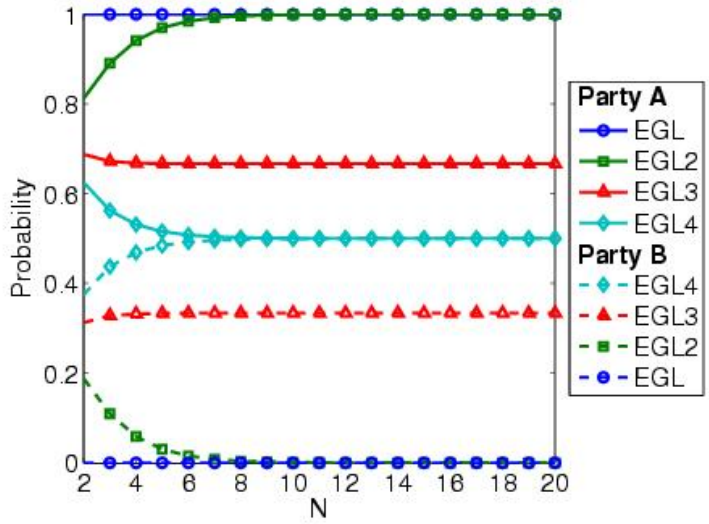
- Consider a PCTL formula $P_{\sim p}$ [ ψ ]
  - if the probability is unknown, how to choose the bound p?
- PRISM allows (for DTMCs) properties of the form $P_{=?}$ [ ψ ]
  - when the outermost operator of a PTCL formula is P
  - "what is the probability that path formula ψ is true?"
  - model checking is no harder: compute the values anyway
- And for MDPs:
  - properties of the form $P_{min=?}$ [ ψ ] and $P_{max=?}$ [ ψ ]
- Experiments: ranges of model/property parameters
  - e.g. $P_{=?}$ [ $F^{\leq T}$ error ] for T=1..100, N=1..5
    where T is a time bound and N is some model parameter
  - identify patterns, trends, anomalies in quantitative results

# Some real PCTL examples

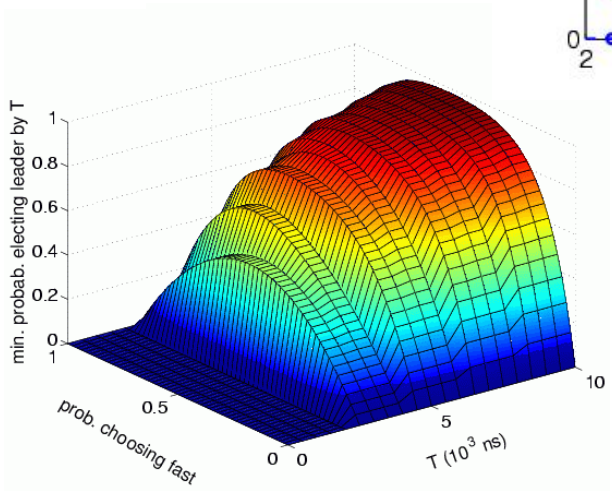
- NAND multiplexing system
  - $P_{=?}$ [ F err/total>0.1 ]
  - "what is the probability that 10% of the NAND gate outputs are erroneous?"
- FireWire communication protocol
  - $P_{=?}$ [ $F^{\leq t}$ done_1 || done_2 ]
  - "what is the probability that a leader node has been elected within t clock-ticks?"
- Security: EGL contract signing protocol
  - $P_{=?}$ [ F (pairs_a=0 & pairs_b>0) ]
  - "what is the probability that the party B gains an unfair advantage during the execution of the protocol?"




Probability that 10% of gate outputs are erroneous for varying gate failure rates and numbers of stages




Probability that parties gain unfair advantage for varying numbers of secret packets sent




Optimum probability of leader election by time T for various coin biases

# Best/worst-case scenarios

- Combining "quantitative" and "exhaustive" aspects
- All possible resolutions of nondeterminism (MDPs)
  - $P_{min=?}$ [ !end2 U end1 ] – "minimum probability of process 1 finishing before process 2, for any scheduling of processes?"
- Computing values for a range of states
  - $P_{=?}$ [ $F^{\leq t}$ reply_count=k {"init"}{min} ]
    "what is the minimum probability, from any initial configuration, that the sender has received k acknowledgements within t clock-ticks?"
  - $P_{=?}$ [ $F^{\leq t}$ elected {tokens≤k}{min} ] – "minimum probability of the leader election algorithm completing within t steps from any state where there are at most k tokens"

# Costs and rewards

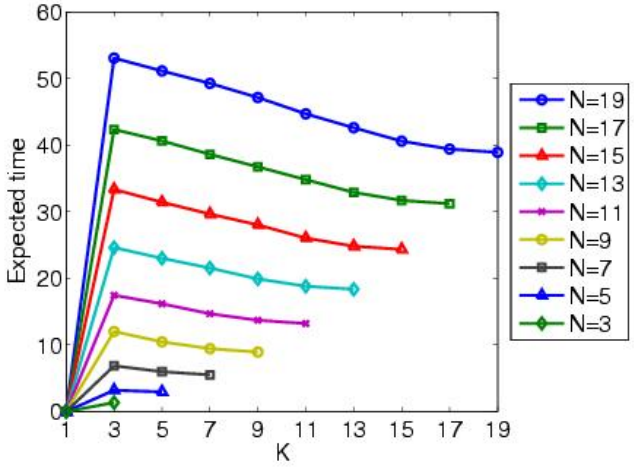
- Augment DTMCs/MDPs with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - no distinction between rewards ("good") and costs ("bad")
  - these can have a wide range of possible interpretations

- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …

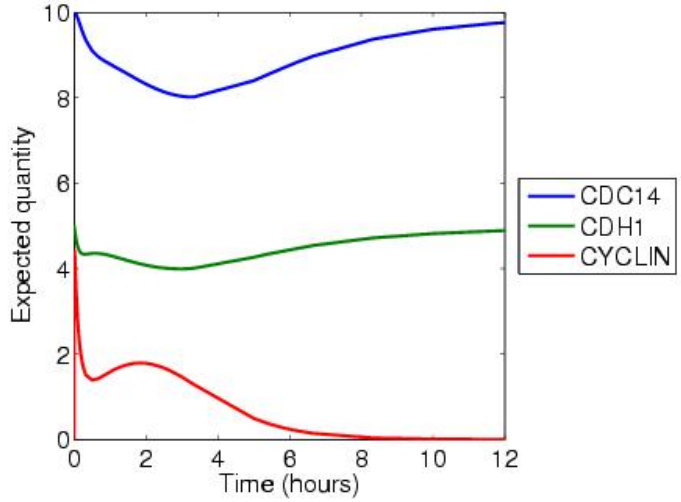
- DTMC/MDP can have (multiple) reward structures $(\underline{\rho},\iota)$
  - state reward function: $\underline{\rho} : S \to \mathbb{R}_{\geq 0}$
  - transition reward function: $\iota : S \times S \to \mathbb{R}_{\geq 0}$ (for DTMCs)
    $\iota : S \times Act \to \mathbb{R}_{\geq 0}$ (for MDPs)

# Cost- and reward-based properties

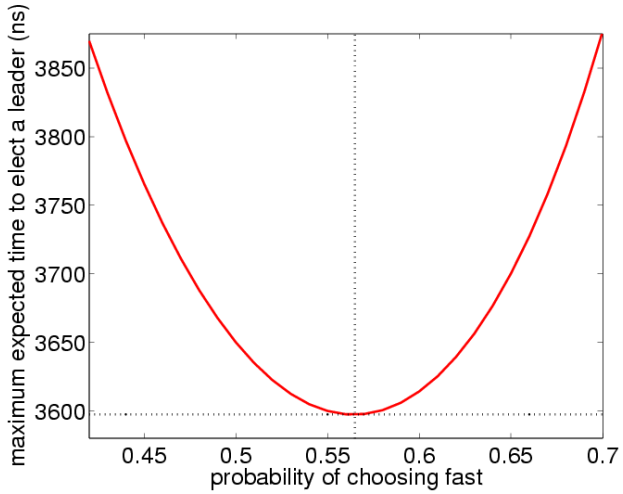
- Formal specification of properties relating to rewards
  - add a new R operator to PCTL, similar to existing P operator
  - reason about expected values of rewards
  - two different interpretations: instantaneous and cumulative
- Instantaneous reward properties
  - state rewards only; state-based measures: “queue size”, “number of open channels”, “concentration of reactant X”, ...
  - $R_{=?}$ [ $I^{=t}$ ] e.g. “expected message queue size at time t?”
- Cumulative reward properties
  - state and transition rewards, e.g. “time”, “power consumption”, “number of messages lost”
  - $R_{=?}$ [ F end ] e.g. “expected time for protocol termination?”
  - $R_{max=?}$ [ $C^{\leq 2}$ ] e.g. “maximum expected power consumption during the first 2 hours that the system is in operation?”




Worst-case expected number of steps to stabilise for initial configurations with K tokens amongst N processes




Expected reactant concentrations over the first 12 hours




Maximum expected time for leader election for various coin biases

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# Probabilistic model checking

- Two distinct phases:
  - model construction, i.e. translation from high-level modelling language (PRISM language, process algebra, …) to DTMC/MDP
  - model checking – qualitative or quantitative PCTL properties
- Probabilistic model checking algorithms
  - graph-based algorithms on underlying transition system
    - reachability, qualitative probabilistic reachability
  - numerical computation – calculation of probabilities, rewards
    - usually, linear equation systems or linear optimisation problems
    - typically use iterative methods, e.g. Gauss-Seidel, value iteration
  - also: simulation-based sampling for approximate analysis
- State-space explosion problem
  - considerable effort devoted to efficient implementations
  - PRISM uses symbolic (BDD-based) data structures, techniques

# Tool support

- Probabilistic model checkers
  - PRISM: DTMCs, MDPs, CTMCs + rewards
  - ETMCC/MRMC: DTMCs, CTMCs + reward extensions
  - LiQuor: LTL verification for MDPs (Probmela language)
  - RAPTURE: prototype for abstraction/refinement of MDPs
  - Simulation-based tools: APMC, Ymer, VESTA
  - CSL model checking for CTMCs: APNN-Toolbox, SMART
  - Multiple formalism/tool solutions: CADP, Möbius
- Tool support provides:
  - high-level languages/formalisms for building models
  - fully automated verification, efficient implementations
  - automation, visualisation of quantitative results
  - discrete event simulations- debugging, approximations

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# PRISM case studies

- Communication and multimedia protocols
  - Bluetooth device discovery, FireWire root contention, IPv4 Zeroconf protocol, IEEE 802.3 CSMA/CD protocol, IEEE 802.11 WiFi wireless LANs, Zigbee (IEEE 802.15.4)

- Security systems/protocols
  - Probabilistic contract signing, Crowds protocol (anonymity), Probabilistic fair exchange, PIN cracking schemes, Negotiation frameworks, Quantum cryptography


- Randomised distributed algorithms for:
  - Byzantine agreement, Consensus, Self-stabilisation, Leader election, Mutual exclusion, …

- Analysis of behaviour/performance/reliability of:
  - Biochemical reaction networks, Dynamic power management, Dynamic voltage scaling algorithms, Manufacturing/control systems, NAND multiplexing, Groupware protocols, …

# Case Study – Contract signing

- Two parties want to agree on a contract
  - each will sign if the other will sign, but do not trust each other
  - in real life, sit down and write signatures simultaneously
  - on an asynchronous network such as the internet?
  - no deterministic protocol without use of a trusted third party

- Partial secret exchange protocol – randomised solution
  - Even, Goldreich, Lempel [EGL85]
  - "viability" – with correct execution of protocol, signatures successfully exchanged
  - "approximate concurrency" – if one party X executes protocol correctly, with very high probability can always obtain signature of Y with approximately same work as Y

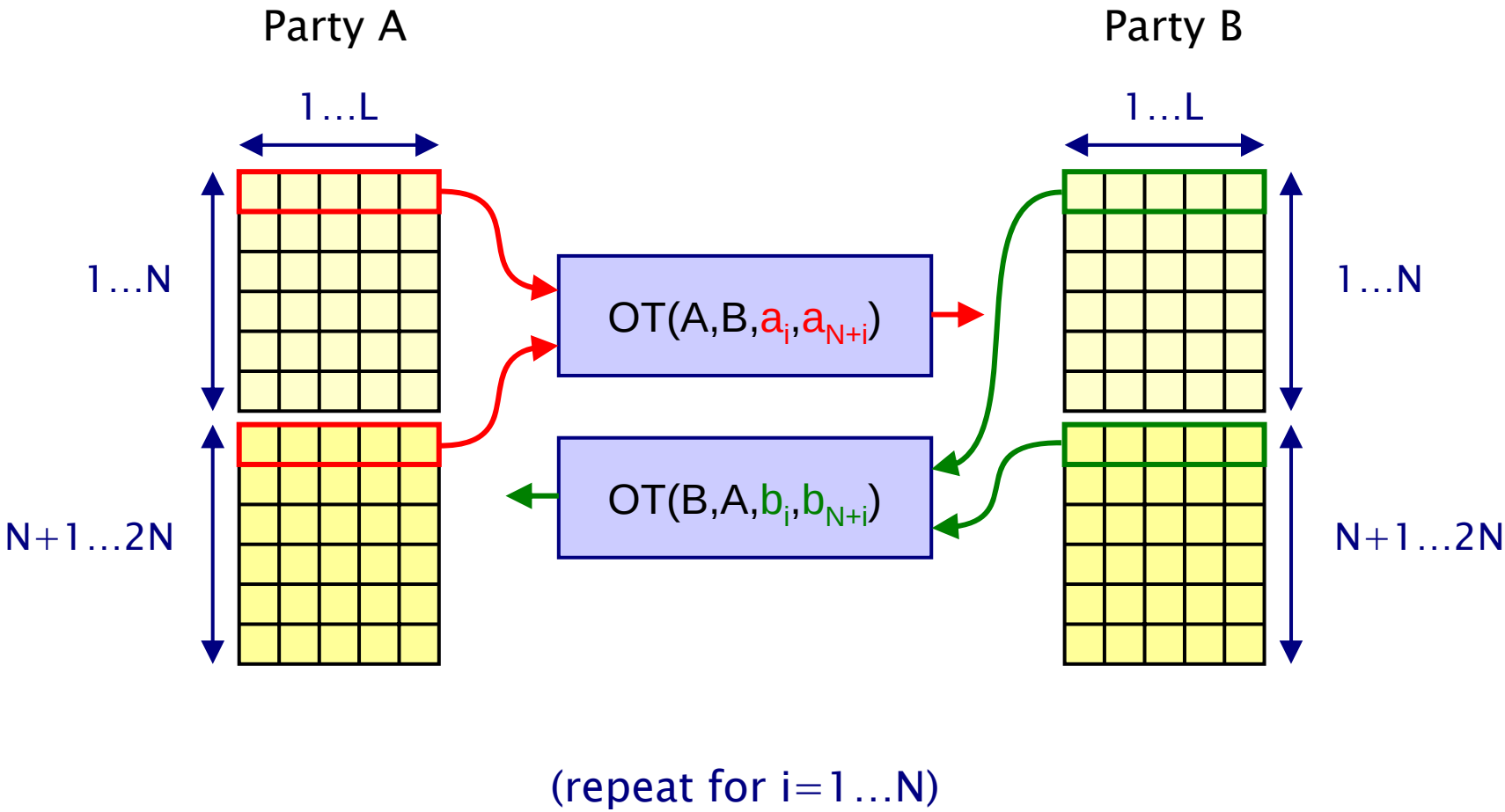
# Contract signing – EGL protocol

- Partial secret exchange protocol for 2 parties (A and B)
- A (B) holds 2N secrets $a_1,\ldots,a_{2N}$ ($b_1,\ldots,b_{2N}$)
  - a secret is a binary string of length L
  - secrets partitioned into pairs: e.g. { $(a_i, a_{N+i})$ | i=1,…,N }
  - A (B) committed if B (A) knows one of A's (B's) pairs
- Uses "1-out-of-2 oblivious transfer protocol" OT(S,R,x,y)
  - S sends x and y to R
  - R receives x with probability ½ otherwise receives y
  - S does not know which one R receives
  - if S cheats then R can detect this with probability ½

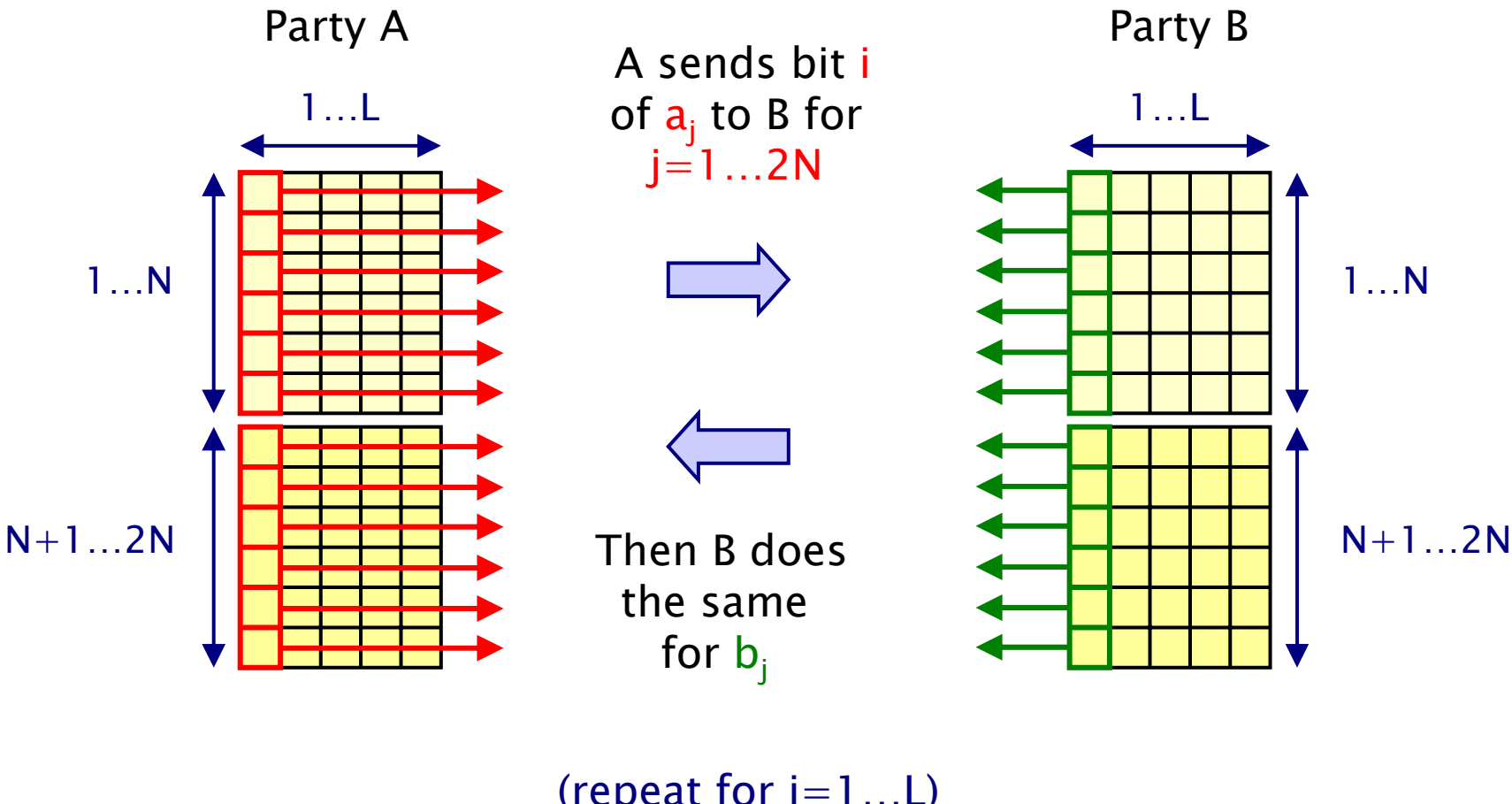
# Contract signing – EGL protocol

(step 1)
for ( i=1,…,N )
    OT( A, B, $a_i$, $a_{N+i}$ )
    OT( B, A, $b_i$, $b_{N+i}$ )
(step 2)
    for ( i=1,…,L )  (where L is the bit length of the secrets)
        for ( j=1,…,2N )
            A transmits bit i of secret $a_j$ to B
        for ( j=1,…,2N )
            B transmits bit i of secret $b_j$ to A

# EGL protocol – Step 1



# EGL protocol – Step 2




(repeat for i=1…L)

# Contract signing – Results

- Modelled in PRISM as a DTMC (no concurrency) [NS06]
- Discovered a weakness in the protocol
  - party B can act maliciously by quitting the protocol early
  - this behaviour not considered in the original analysis
- PRISM analysis shows
  - if B stops participating in the protocol as soon as he/she has obtained one of A's pairs, then, with probability 1:
    - B will end up possessing a pair of A's secrets
    - A will not have complete knowledge of any pair of B's secrets
  - protocol is not fair under this attack:
    - B has a distinct advantage over A

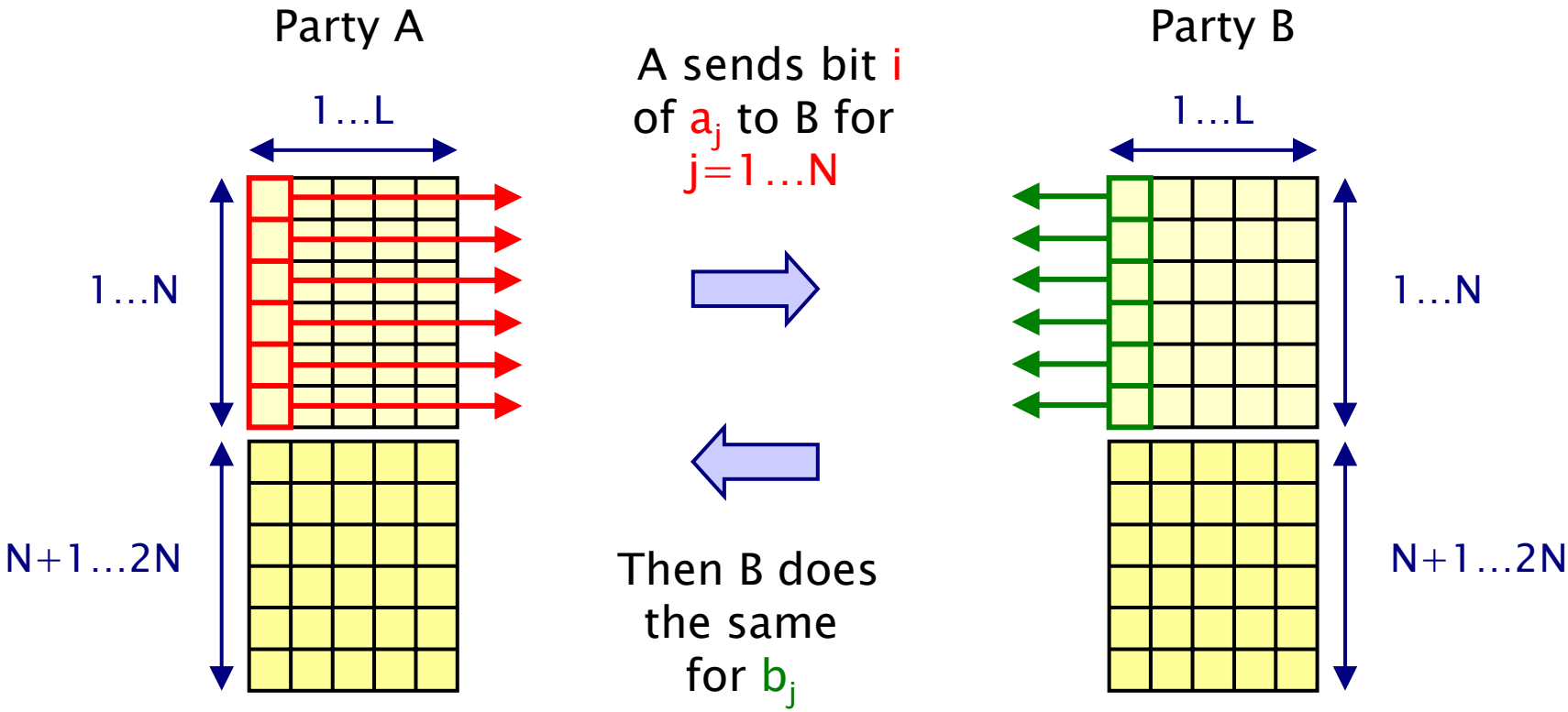
# Contract signing – Results

- The protocol is unfair because in step 2:
  - A sends a bit for each of its secret before B does
- Can we make this protocol fair by changing the message sequence scheme?
- Since the protocol is asynchronous, the best we can hope for is:
  - B (or A) has this advantage with probability ½
- We consider 3 possible alternative message sequence schemes…

# Contract signing – EGL2

(step 1)
…
(step 2)
for ( i=1,…,L )
  for ( j=1,…,N )  A transmits bit i of secret $a_j$ to B
  for ( j=1,…,N )  B transmits bit i of secret $b_j$ to A
  for ( j=N+1,…,2N )  A transmits bit i of secret $a_j$ to B
  for ( j=N+1,…,2N )  B transmits bit i of secret $b_j$ to A

# Modified step 2 for EGL2

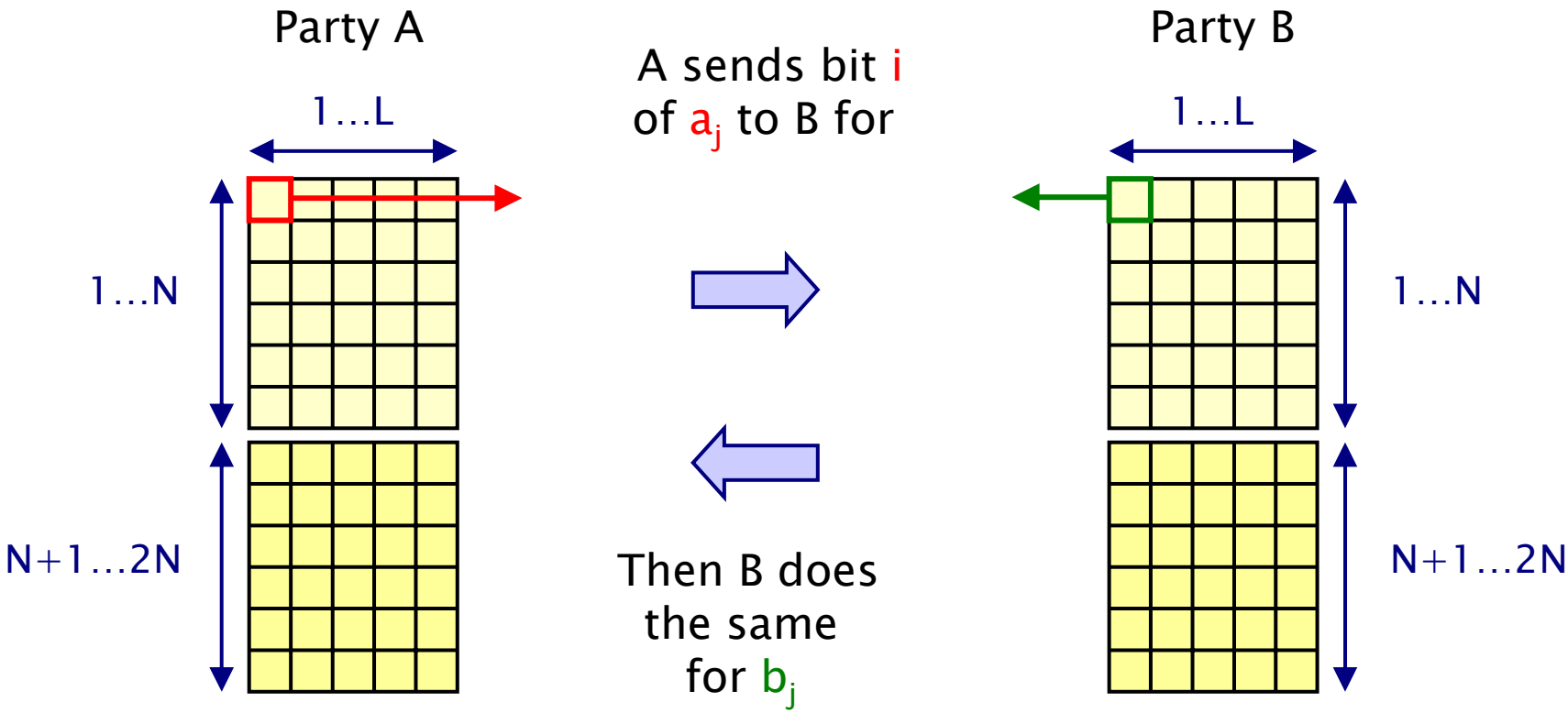



(after j=1…N, send j=N+1…2N)
(then repeat for i=1…L)

# Contract signing – EGL3

(step 1)
…
(step 2)
**for** ( i=1,…,L ) **for** ( j=1,…,N )
A transmits bit i of secret $a_j$ to B
B transmits bit i of secret $b_j$ to A
**for** ( i=1,…,L ) **for** ( j=N+1,…,2N )
A transmits bit i of secret $a_j$ to B
B transmits bit i of secret $b_j$ to A

# Modified step 2 for EGL3

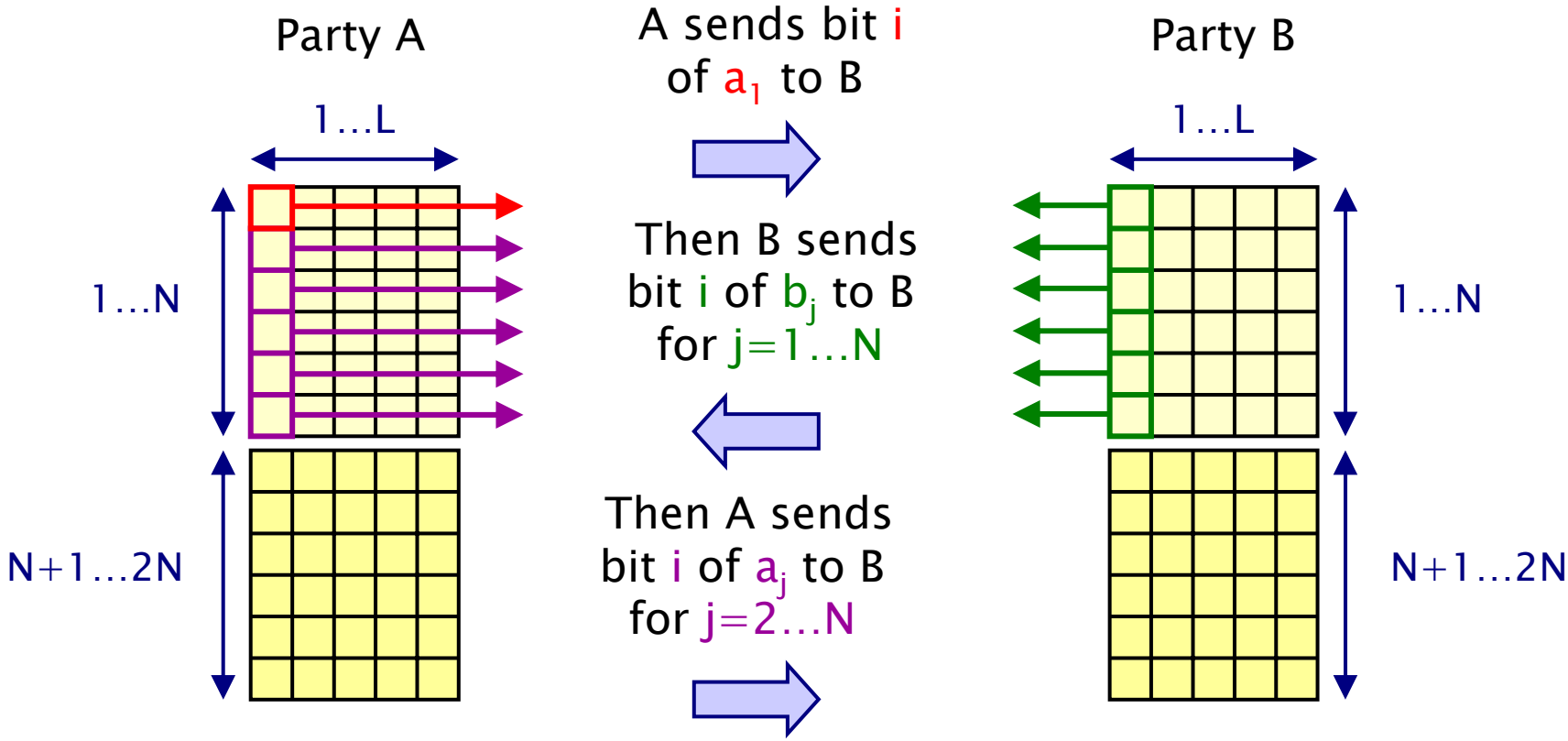



(repeat for j=1…N and for i=1…L)
(then send j=N+1…2N for i=1…L)

# Contract signing – EGL4

(step 1)
…
(step 2)
for ( i=1,…,L )
  A transmits bit i of secret $a_1$ to B
    for ( j=1,…,N )  B transmits bit i of secret $b_j$ to A
    for ( j=2,…,N )  A transmits bit i of secret $a_j$ to B
for ( i=1,…,L )
  A transmits bit i of secret $a_{N+1}$ to B
    for ( j=N+1,…,2N )  B transmits bit i of secret $b_j$ to A
    for ( j=N+2,…,2N )  A transmits bit i of secret $a_j$ to B

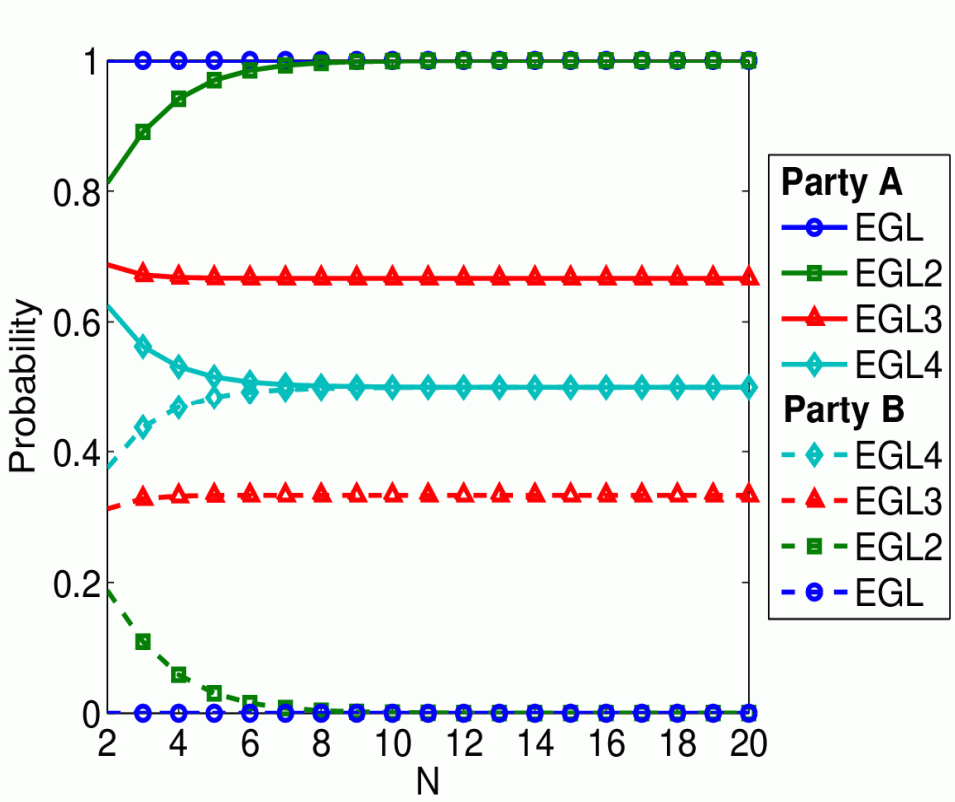
# Modified step 2 for EGL4




(repeat for i=1…L)
(then send j=N+1…2N in same fashion)

# Contract signing – Results

- The chance that the protocol is unfair
  - probability that one party gains knowledge first
  - $P_{=?}[F\ know_B \wedge \neg know_A]$ and $P_{=?}[F\ know_A \wedge \neg know_B]$



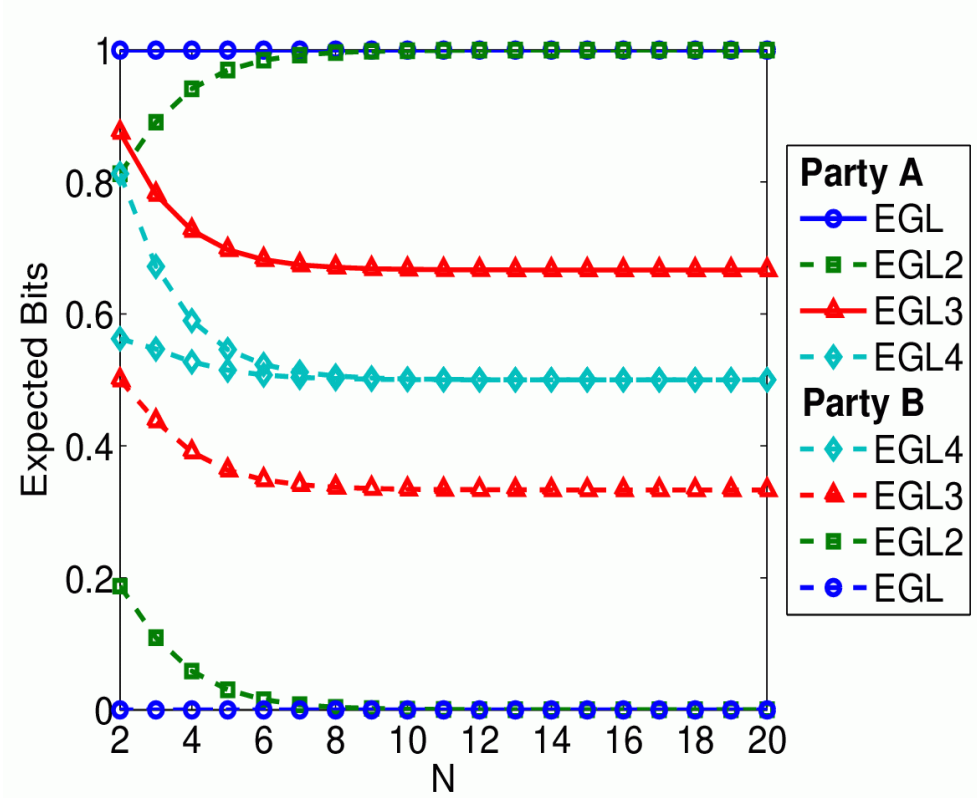
# Contract signing – Results

- How unfair the protocol is to each party
  - expected number of bits that a party needs to know a pair once the other party knows a pair
  - need to modify the model and define a reward structure
  - dependent on which party we are considering

- Expected number of bits that A needs to know a pair once B knows a pair
  - add a transition to a new state labelled by “done” as soon as B knows a pair
  - assign a reward equal to the number of bits that A requires to know a pair to this transition
  - check the formula $R_{=?}$[F done]

# Contract signing – Results

- How unfair the protocol is to each party
  - expected number of bits that a party needs to know a pair once the other party knows a pair



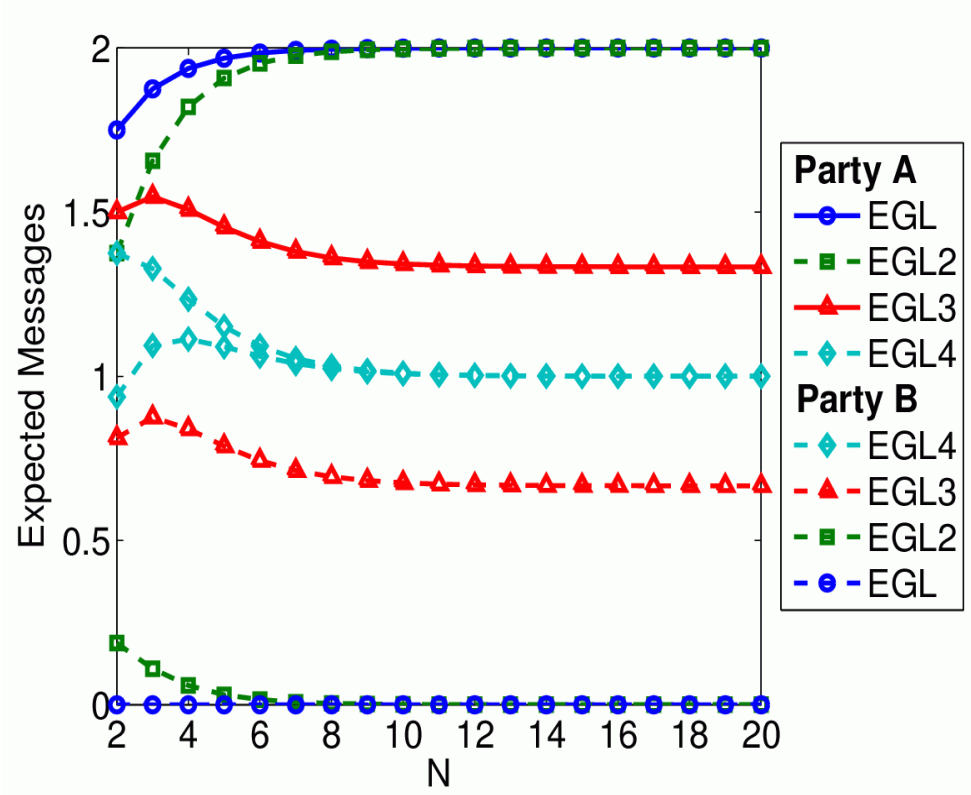
# Contract signing – Results

- The influence that each party has on the fairness
  - once a party knows a pair, the expected number of messages from this party required before the other party knows a pair
  - measures the influence as a corrupted party can delay its messages
  - need to define a reward structure
  - dependent on which party we are considering

- Once B knows a pair, the expected number of messages from B required before A knows a pair
  - assign reward of 1 to transitions which correspond to B sending a message to A from a state where B knows a pair
  - check the formula $R_{=?}[F\ know_A]$

# Contract signing – Results

- The influence that each party has on the fairness
  - once a party knows a pair, the expected number of messages from this party required before the other party knows a pair



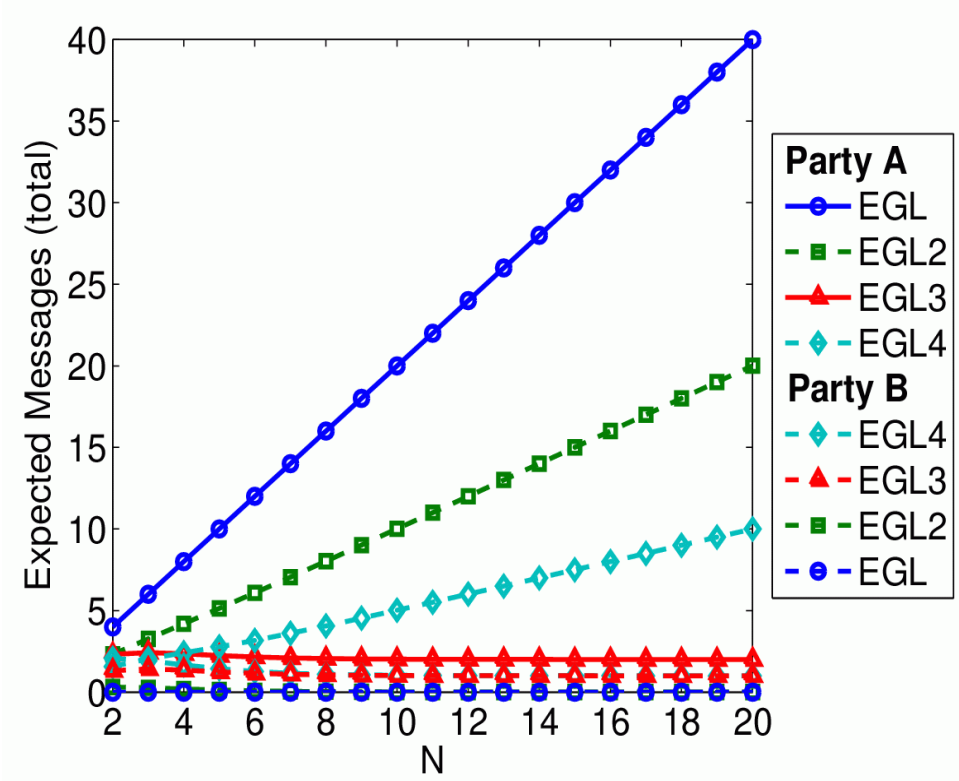
# Contract signing – Results

- The duration of unfairness of the protocol
  - once a party knows a pair, the expected total number of messages that need to be sent (by either party) before the other knows a pair
  - need to define a reward structure
  - dependent on which party we are considering

- Once B knows a pair, the expected total number of messages that need to be sent before A knows a pair
  - assign reward of 1 to transitions which correspond to either party sending a message from a state where B knows a pair
  - check the formula $R_{=?}[F\ know_A]$

# Contract signing – Results

- The duration of unfairness of the protocol
  - once a party knows a pair, the expected total number of messages that need to be sent before the other knows a pair

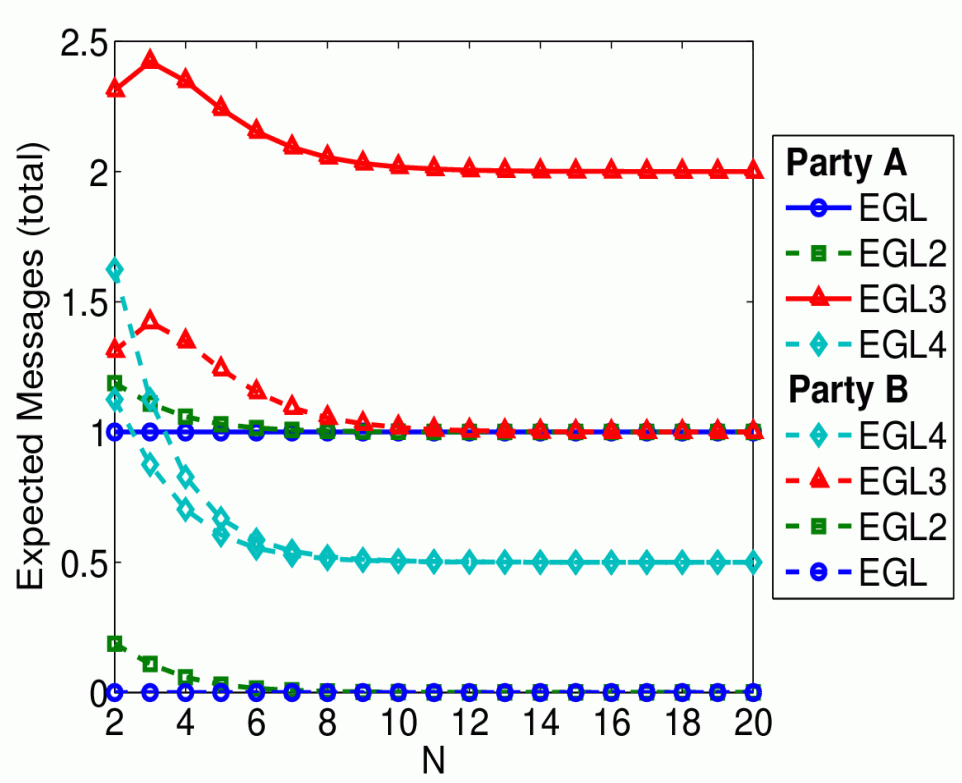


# Contract signing – Results

- Results show EGL4 is the 'fairest' protocol
- Except for duration of fairness measure…
- Expected messages that need to be sent for a party to know a pair once the other party knows a pair
  - this value is larger for B than for A
  - in fact, as N increases, this measure increases for B and decreases for A
- Solution
  - if a party sends a sequence of bits in a row (without the other party sending messages in between), require that the party send these bits as as a single message

# Contract signing – Results

- The duration of unfairness of the protocol
  - once a party knows a pair, the expected total number of messages that need to be sent before the other knows a pair



# Summary

- Summary
  - probabilistic model checking
  - probabilistic models: DTMCs, MDPs, …
  - property specifications, implementation and tool support
  - case studies, probabilistic contract signing

- For more information, see the PRISM web page
  - www.prismmodelchecker.org
  - 11 part lecture course
  - related publications, talks, tutorials, links
  - on-line example repository (40+ case studies)
  - PRISM tool download: binaries, source code (GPL)
  - on-line manual, tutorial