Duality and the completeness of the modal μ-calculus

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Abstract

We consider the modal μ-calculus due to Kozen, which is a finitary modal logic with least and greatest fixed points of monotone operators. We extend the existing duality between the category of modal algebras with homomorphisms and the category of descriptive modal frames with contractions to the case of having fixed point operators. As a corollary, we obtain completeness results for two proof systems due to Kozen (finitary and infinitary) with respect to certain classes of modal frames. The rules are sound in every model, not only for validity.

1 Introduction

When writing programs, it is often useful to think operationally in terms of 'what happens next'. This intuition is provided by the *operational* semantics $\mathcal{O}$. However, the operational semantics is not always the most natural description of a program, and an alternative, *denotational*, view is also used. Here programs are elements of some abstract domain $\mathcal{D}$, and operations on programs are defined as functions on them. Domains are often defined by means of recursive domain equations solved in some category, *e.g.* SFP domains or ultrametric spaces. Then, the statement of full abstractness establishes a strong correspondence between the two views, and can be understood as means for switching between the operational and denotational intuitions without loss of information.

Recently, Abramsky [1] has argued that this framework should be extended to include program logics (the *logical* view $\mathcal{L}$) as well. By means of a program logic one can express properties of programs, such as safety and liveness. As previously, a statement of strong correspondence with the other view is called for. This is provided by *Stone duality* ($\mathcal{L}^{op} \simeq \mathcal{D}$), *i.e.* a contravariant equivalence of the respective categories. Abramsky considers domain equations solved in the category of SFP; he defines a category of domain prelocales

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(coherent propositional theories) with type constructors on them analogous to the type constructors on SFP domains, so one can "mechanically" unpack a given domain equation to obtain the corresponding logic (up to isomorphism).

On the one hand, Abramsky's framework is very general, since it can deal with domain equations involving the function space and the Plotkin powerdomain constructors. On the other hand, its expressive power, being essentially the logic of compact opens, and hence simply a distributive lattice of a certain kind, can be considered restrictive. For example, in the flat domain of natural numbers with the standard topology, one cannot express the property *even* (the set of even numbers). A possible extension allowing richer structure, already suggested by Abramsky [1], would involve adding fixed point operators. Thus, the open set of even numbers can be denoted by $\mu x.0 \vee s(s(x))$, *i.e.* the least fixed point of the (monotone) operator $\lambda x.0 \vee s(s(x))$, where $s(n)$ denotes the successor function.

The fixed point operators are particularly important in connection with program properties, in particular *fairness* properties. An early result, due to Park [18], concerns the setting of languages of finite and ω-infinite words over some alphabet, in which it is shown that fairmerge (a multifunction which produces interleavings of two words in such a way that the whole of each word is taken) can be expressed as an alternation of the greatest and least fixed points of certain operators monotone in the complete lattice of infinitary word languages, but that neither of the two fixed points suffices on its own. More recent work due to Larsen [15] shows that fixed points can be used to express safety and liveness properties.

Fairness properties are statements of substantial complexity. Intuitively, (strong) fairness can be expressed as $\forall P$. *P infinitely often enabled implies P infinitely often taken*, where P is a process (we refer the reader to *e.g.* [5, 14] for more detail). Fairness is known to cause many difficulties: it requires the addition of infinite behaviours, enforces unbounded delay, destroys the continuity of semantic operations with respect to standard topologies, and introduces the need for transfinite induction. Our reasons for working with fairness are motivated by the fact that it is important in practice, particularly in communication protocols and distributed systems, and that certain properties of programs *cannot* be proved unless fairness is assumed. Moreover, there is evidence in the form of concrete software systems that fairness can be expressed in logics such as the temporal logic CTL*, and automatically verified [3].

As a concrete system which is capable of expressing fairness, we adopt the modal μ-calculus due to Kozen [12]. It is essentially a finitary, propositional modal logic with least and greatest fixed points of monotone operators. As a modal logic, it fits in better with "domain theory in logical form" than the (state-based) temporal logics such as CTL*. Models for the calculus are labelled transition systems; the closed formulas describe sets of states, and the open formulas can be viewed as maps on the powerset of states. These maps are monotone but not necessarily order-continuous because there exist models and formulas involving fixed points whose interpretation in these models requires unfoldings for ordinals higher than ω. The appealing features of the modal μ-calculus are: its decidability (which should be contrasted with the infinitary Hennessy-Milner logic), and the expressive power (for example, it is strictly more expressive than PDL and CTL*). To give some intuition, the following modal μ-calculus formulas have the corresponding meaning in any transition system model:

1. $\mu x.[a]x$ (every a-path is finite);

2. $\nu x.\langle a\rangle x$ (there exists an infinite a-path);

3. $\mu x.\varphi \vee ([a]x \wedge \langle a \rangle \mathbf{tt})$ (along every a-path eventually φ holds – liveness);

4. $\nu x.\langle a \rangle \mathbf{tt} \wedge [L]x$ (a is always possible – safety);

5. $\nu z.\mu y.(\langle a \rangle \mathbf{tt} \vee [L]y) \wedge [L]z \wedge \langle L \rangle \mathbf{tt}$ (a is possible infinitely often);

6. $\nu z.\mu y.[L - a]y \wedge [L]z \wedge \langle L \rangle \mathbf{tt}$ (a happens infinitely often)

where L denotes some (finite) set of actions $\{a_1, \ldots, a_n\}$, and $[L]\varphi$ is taken to mean $[a_1]\varphi \wedge \ldots \wedge [a_n]\varphi$ (with the dual interpretation for $\langle L \rangle$).

One long-standing problem with the modal μ-calculus is the lack of a satisfactory completeness statement (satisfactory in the sense of being sound over extensions to the logic), although we should point out that a number of weaker, but workable, systems are in existence, *e.g.* [12, 23]. Although known to be decidable, see *e.g.* [13], the modal μ-calculus has not lent itself to an axiomatic characterization. The original calculus introduced by Kozen [12] is not yet known to be complete, although the subcalculus of *aconjunctive* formulas is shown to be so in [12]. Kozen obtains a completeness result in [13] by introducing an infinitary proof rule. Walukiewicz [23] obtains a similar result with the addition of an induction schema. His rules have the property that if all the premises are valid then the conclusion is always valid, but not the more intuitive property that, for any model $\mathcal{M}$, if all the premises are true in $\mathcal{M}$ then the conclusion must be true in $\mathcal{M}$ (as Walukiewicz himself demonstrates by a suitable example).

As Walukiewicz points out, the question of a complete axiomatization is of interest even though the calculus is known to be decidable, since an axiomatization gives us deeper insight into the calculus, and furthermore would enable us to construct a proof assistant to mitigate the complexity problems posed by an automated decision procedure carrying out an exponential task.

The aim of the paper is to cast light on this problem by deriving a suitable duality theory for the modal μ-calculus. Although the presentation of the full picture must await technical details to be presented later, it may be helpful to describe at this stage the contribution made by duality theory.

Traditionally completeness results for modal systems have often used the construction of a *canonical Kripke model* for the system in question. However, in the case of the μ-calculus, This is not possible. The reason is that any attempt to construct such a model generates among the possible worlds of the model the so-called *non-standard ultrafilters* (on which see further Section 10). If we think loosely of a fixed point formula as a limit of a sequence of finite approximations, then the non-standard ultrafilters are those worlds where the fixed point formula holds without any of the approximations holding. The problem with the Kripke model construction is that any successful interpretation of a fixed point formula must map it onto a set which includes the non-standard ultrafilters, whereas in the Kripke interpretation the meets and joins which are used to model such formulas are simply intersections and unions respectively of the sets representing their approximations, with the result that the non-standard ultrafilters are not guaranteed to belong to the appropriate interpretations.

In our extension of modal duality theory, on the other hand, we circumvent this problem by restricting the possible interpretations of formulas in the μ-calculus to a subalgebra of the powerset algebra on the set of ultrafilters. In those cases where some non-standard ultrafilter would not be present in the union, say, of a set of approximations to the interpretation of a fixed point formula, the union does not belong to our subalgebra, but the non-standard ultrafilter will be present in the join in the subalgebra (which is, in topological terms, the closure of the union).

Duality theory originates from Stone, who set out to formulate a representation theorem for Boolean algebras as a field of sets so that the operations of conjunction, disjunction and negation can be modelled by their set-theoretic counterparts. Stone proposed a construction of a topological space $\mathsf{Spec}B$ from an arbitrary Boolean algebra B taking as points the ultrafilters of B (denoted $U(B)$) and the topology given by the basis:

$$U_a \quad = \quad \{U \in U(B) \mid a \in U\}.$$

This gives rise to isomorphisms:

$$\begin{array}{lcll} B & \cong & \mathsf{ClopSpec}B & \beta : b \mapsto \{U \in U(B) \mid b \in U\} \\ S & \cong & \mathsf{SpecClop}S & \gamma : s \mapsto \{X \in \mathsf{Clop}S \mid s \in X\} \end{array}$$

where S is any compact, Hausdorff and totally disconnected space, subsequently named a *Stone space*. It was observed later that this correspondence can be expressed as an equivalence of categories ($\mathsf{Bool}^{op} \simeq \mathsf{Stone}$); for more detail see [10].

We take an existing duality for modal algebras [19], and extend it to the case of having fixed point operators. We should point out that, until now, there was no duality statement for a setting with fixed points, as the main focus is on algebraic structures. A modal algebra is a Boolean algebra B together with a unary operation τ (corresponding to $\Box$) which preserves finite meets. Modal algebras have (descriptive) modal frames as their duals [7]. In the notation of [19], we have $M \cong (M_*)^*$ for any modal algebra, where the dual frame of M is denoted M_*, and the dual algebra of a frame F is F^*. One can think of modal algebras as certain arrows in the category Bal of Boolean algebras and hemimorphisms (*i.e.* maps preserving finite meets) between them. This category admits a duality theorem which is more general than the traditional Stone duality, in the sense that it is a duality with the category of spaces with continuous relations in the sense defined in [19], rather than continuous functions.

The paper is organized as follows. Section 2 gives a summary of the notation and definitions, Section 3 introduces the (finitary and infinitary) modal μ-calculi denoted L_μ and L_μ^{inf} respectively, and Section 4 discusses completeness of algebraic and relational semantics for modal logics. In Section 5, following some ideas due to Niwinski [17], we we define the notion of a *modal μ-algebra*, which is a modal algebra in which the fixed points of certain monotone operators are required to exist. The proposed structure is, in general, weaker than the complete lattices of [17] and [12], since only the infinite joins/meets *definable* by means of a fixed point operator must exist. We consider two special cases of modal μ-algebras: of the *first kind*, in which least fixed points are given as the meet of the pre-fixed-points, and of the *second kind*, in which they correspond to the join of the transfinite chain of unfoldings. The two characterizations of fixed points coincide on complete lattices, but *not* in weaker structures. We show that each of the modal μ-algebras mentioned above corresponds to *different* proof rules, a fact reflected in the respective Lindenbaum algebras: Kozen's finitary rules (μL) and (μR) correspond to the algebras of the first kind, while the infinitary rule (inf) to the algebras of the second kind (more precisely, a subclass called the second ω-kind). Section 6 introduces *modal μ-frames*, the duals of modal μ-algebras.

In Section 7, we examine the representation theorem of modal algebras in the case of modal μ-algebras. The modal frames that arise through this construction (descriptive modal μ-frames) have as bases families of clopens closed under the interpretation of fixed points; in other words, the interpretation of a fixed point term in the Stone space of a modal μ-algebra is a clopen subset. As a corollary of the representation theorem, in Section 8.1 we obtain canonical models for Kozen's calculi L_μ and L_μ^{inf}, and hence *completeness* results with respect to certain classes of modal μ-frames. Finally, in Section 9 we derive a duality theory for the respective categories of modal μ-algebras and the (dual) descriptive modal μ-frames. We remain essentially within the framework of the original Stone duality, in the sense that given a modal μ-frame $\langle X, r, T\rangle$, the dual of some modal μ-algebra $\langle B, \tau\rangle$, the underlying space of (X, T) is a Stone space.

2 Notation and preliminaries

Some familiarity with Boolean algebras, modal logics, and topology is assumed.

Let X be a topological space. X is *totally disconnected* iff the only connected subsets are the singleton sets ($Y \subseteq X$ is *connected* iff the only clopen sets in the subspace topology are $\emptyset$ and Y). X is *totally separated* iff every pair of distinct points can be separated by a clopen subset, *i.e.* given x, y there exists a clopen set C such that $x \in C$ and $y \notin C$. A space is a *Stone space* iff it is compact, Hausdorff and totally disconnected, or, equivalently, compact and totally separated.

By a *field of subsets* we shall mean a family of subsets closed under the set-theoretic operations of intersection, union and complement. A field of subsets $\mathcal{F}$ is *reduced* iff every pair of distinct points is separated by a set $A \in \mathcal{F}$, and *perfect* iff every ultrafilter of $\mathcal{F}$ is determined by a point (*i.e.* it is of the form $\{A \in \mathcal{F} \mid x \in A\}$ for some x). F is a perfect and reduced field of subsets of X if and only if the topology on X defined by taking F as a clopen basis is a Stone space.

The *Stone space* of a Boolean algebra B is defined by taking the set of all ultrafilters $U(B)$ as points, and the family of all sets $\beta(b) = \{U \in U(B) \mid b \in U\}$ as the clopen basis.

We recall two well-known characterizations of least and greatest fixed-points of monotone functions on a complete lattice. For an overview of fixed point theorems see *e.g.* [16]. The following characterization is as the meet or join of the set of *pre-fixed-points*.

Lemma 2.1 (Attributed to Knaster & Tarski) *Let $(L, \leq)$ be a complete lattice. Then every monotone function $f : L \longrightarrow L$ has the least fixed point:*

$$\mu x.f(x) \quad = \quad \bigwedge \{a \in L \mid f(a) \leq a\}$$

and its dual, the greatest fixed point:

$$\nu x.f(x) \quad = \quad \bigvee \{a \in L \mid a \leq f(a)\}.$$

This first characterization gives us no direct way to calculate the fixed points, which has prompted several authors to formulate the following alternative characterization.

Lemma 2.2 (Attributed to Hitchcock & Park) *Let $(L, \leq)$ be a complete lattice, $f : L \longrightarrow L$ a monotone function. Define the (increasing) chain of unfoldings of f by:*

$$\begin{array}{rcll} f^0(0) & = & 0 & \\ f^{\alpha+1}(0) & = & f(f^\alpha(0)) & \alpha \textit{ successor ordinal} \\ f^\lambda(0) & = & \bigvee \{f^\alpha(0) \mid \alpha < \lambda\} & \lambda \textit{ limit ordinal.} \end{array}$$

Then there exists an ordinal α of cardinality less than or equal to that of L such that $\beta \geq \alpha$ implies

$$\mu x.f(x) \quad = \quad f^{\beta}(0).$$

Dually, define the (decreasing) chain of unfoldings of f by:

$$\begin{array}{rcll} f_0(1) & = & 1 & \\ f_{\alpha+1}(1) & = & f(f_{\alpha}(1)) & \alpha \text{ successor ordinal} \\ f_{\lambda}(1) & = & \bigwedge \{f_{\alpha}(1) \mid \alpha < \lambda\} & \lambda \text{ limit ordinal.} \end{array}$$

Then there exists an ordinal α of cardinality less than or equal to that of L such that $\beta \geq \alpha$ implies

$$\nu x.f(x) \quad = \quad f_{\beta}(1).$$

We say that these lemmas characterize fixed points of the *first* and *second kind* respectively. It should be noted that the two characterizations coincide on complete lattices, but not, in general, on weaker structures. We confirm this by considering

Example 2.3 Let L be the distributive lattice with elements $\{x_i \mid i \in \omega\} \cup \{y_i \mid i \in \omega\}$ satisfying $x_i < x_j$ and $y_i > y_j$ for $i < j$, and $x_i < y_j$ for all $i, j \in \omega$. Let $f : L \longrightarrow L$ be defined

$$\begin{array}{rcl} f(x_i) & = & f(x_{i+1}) \\ f(y_i) & = & \left\{ \begin{array}{ll} y_0 & \text{if } i = 0 \\ y_{i-1} & \text{otherwise} \end{array} \right. \end{array}$$

This is a monotone function with least fixed point at y_0. The fixed point is of the first kind because it is the only pre-fixed point, but it is not of the second kind because the ω unfolding of f does not exist, the x_i having no least upper bound in L.

Thus we can have ordered structures weaker than complete lattices where fixed points of the first kind fail to be of the second kind. However all fixed points of the second kind are of the first kind:

Proposition 2.4 *A least fixed point of the second kind is also of the first kind.*

Proof: Let L be a poset, and let $f : L \longrightarrow L$ be a monotone function and suppose that the unfoldings $f^{\alpha}(0)$ exist for all ordinals α so that the least fixed point of f is given by $f^{\gamma}(0)$ where γ is the closure ordinal.

Let y be a pre-fixed point of f. We show that $f^{\alpha}(0) \leq y$ for all ordinals α by transfinite induction.

- The base case is simply $f^0(0) = 0 \leq y$.
- If $f^{\alpha}(0) \leq y$ then $f^{\alpha+1}(0) = f(f^{\alpha}(0)) \leq f(y) \leq y$ by the monotonicity of f.
- If λ is a limit ordinal and $f^{\alpha}(0) \leq y$ for all $\alpha < \lambda$ then

$$f^{\lambda}(0) = \bigvee \{f^{\alpha}(0) \mid \alpha < \lambda\} \leq y$$

Thus, the least fixed point $f^{\gamma}(0)$ is a lower bound for the set of prefixed points and hence must be the meet because it is itself a pre-fixed point. $\square$

3 Modal μ-calculus

We take Act to be a fixed finite set of *actions*. The set L_μ of terms of the modal μ-calculus is defined inductively as follows. Suppose we are given a countable set of variables $V = \{x_0, x_1, \ldots\}$. The μ*-terms* are:

$V \subseteq L_\mu$	(variables)
$\mathbf{tt}, \mathbf{ff} \in L_\mu$	(constants)
$t, t' \in L_\mu \Rightarrow t \wedge t', \neg t, [a]t \in L_\mu$	$a \in \text{Act}$
$t \in L_\mu,\ x \in V,\ x$ positive in $t \Rightarrow \mu x.t \in L_\mu$	

where x *is positive in* t is a syntactic restriction defined as x occurring within an even scope of negations. This restriction implies monotonicity when viewing (open) terms as maps on formulas. The connectives $\vee$ and $\langle a \rangle$ are the usual duals to $\wedge$ and $[a]$ respectively. In particular,

$$\langle a \rangle t =_{\text{def}} \neg[a]\neg t.$$

The variable-binding operator μ denotes the least fixed point. The greatest fixed point ν is defined as the dual of the least fixed point as follows:

$$\nu x.t =_{\text{def}} \neg\mu y.\neg t(x := \neg y).$$

Here, and in general throughout the paper, $\varphi(x := \psi)$ refers to the result of substituting ψ for all free occurrences of x in φ.

We give a sequent style presentation of the system due to Kozen [12]. The structural rules and rules for the basic connectives are those for Gentzen's system **LK** for classical logic [6]. We augment these with the rules for minimal modal action logic:

$$\frac{\{x \mid [a]x \in \Gamma\}, \varphi \vdash \{y \mid \langle a \rangle y \in \Delta\}}{\Gamma, \langle a \rangle \varphi \vdash \Delta}\,(\Diamond L) \qquad \frac{\{x \mid [a]x \in \Gamma\} \vdash \varphi, \{y \mid \langle a \rangle y \in \Delta\}}{\Gamma \vdash [a]\varphi, \Delta}\,(\Box R)$$

(the latter being the dual of the former), together with the following rules for the μ- and ν-terms (strictly speaking, since ν is a dual of μ, the rules for ν are not necessary; we include them here for the sake of clarity):

$$\frac{\varphi(x := \psi) \vdash \psi}{\mu x.\varphi \vdash \psi}\,(\mu L) \quad x \text{ not free in } \psi \qquad \frac{\Gamma \vdash \varphi(x := \mu x.\varphi), \Delta}{\Gamma \vdash \mu x.\varphi, \Delta}\,(\mu R)$$

$$\frac{\psi \vdash \varphi(x := \psi)}{\psi \vdash \nu x.\varphi}\,(\nu R) \quad x \text{ not free in } \psi \qquad \frac{\Gamma, \varphi(x := \nu x.\varphi) \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta}\,(\nu L)$$

We denote the above system by L_μ. In the light of the two characterizations of fixed points mentioned in Section 2, the rule (μL) can be explained as stating that the term $\mu x.\varphi$ is a *lower bound* for pre-fixed-point terms. This can be seen easily by viewing the (open) term φ as a map on terms; then $\varphi(x := \psi)$ can be read as the application of the map φ to the argument ψ, and $\varphi(x := \psi) \vdash \psi$ as "ψ is a pre-fixed-point of the map φ", and the full rule as "if ψ is a pre-fixed-point of the map φ then $\mu x.\varphi$ is below ψ". By applying a similar argument, the rule (μR) can be read as stating $\mu x.\varphi$ is a *pre-fixed-point* of the map φ. It would thus appear that this system of rules corresponds to the characterization of the least fixed point as the meet of pre-fixed-points, a fact reflected in its Lindenbaum algebra, as we show later.

In order to reflect the alternative characterization, one would expect an inductive rule, but it is not yet clear what this would be for arbitrarily large closure ordinals. In the case where the closure ordinal is ω, the following infinitary rule due to Kozen [13] is appropriate:

$$\frac{\varphi^i \vdash \psi, \text{ all } i \in \omega}{\mu x.\varphi \vdash \psi}\ (\text{inf})$$

where φ^i denotes the i-th unfolding of the term φ defined by:

1. $\varphi^0 = 0$
2. $\varphi^{i+1} = \varphi(x := \varphi^i)$

This rule states that $\mu x.\varphi$ is the ω-*join* of the set of the i-th unfoldings of φ for $i \in \omega$. We denote the system L_μ plus the rule (inf) by L_μ^{inf}.

4 Modal algebras, modal frames and completeness

We use the framework of modal duality as originally proposed by Halmos [8], Goldblatt [7] and later developed by Sambin and Vaccaro [19] and extend this to give an account of the fixed point operators.

Given any modal logic L with simple modalities $\Box$ and $\Diamond$ satisfying the analogue of rules ($\Diamond L$) and ($\Box R$) above, one can give (algebraic) semantics to L in terms of a *modal algebra* as defined below.

Definition 4.1 *A* modal algebra [19] *is a pair $M = \langle B, \tau \rangle$ where B is a Boolean algebra and τ is an operation satisfying:*

$$\begin{array}{lrcl} (A1) & \tau 1 & = & 1 \\ (A2) & \tau(a \wedge b) & = & \tau a \wedge \tau b. \end{array}$$

One can now state completeness of algebraic semantics as follows:

> "for every logic L and every formula φ, $\vdash_L \varphi$ iff φ is valid in every modal algebra satisfying the axioms of L".

The alternative, relational, semantics for the logic L is based on the notion of a Kripke frame, that is, a pair $\langle X, r \rangle$ where X is a set of possible worlds and r is a binary relation of accessibility between worlds. Propositions are modelled by sets of possible worlds (subsets of X), the classical connectives are modelled by their set-theoretic counterparts and the $\Box$-modality is modelled by the operation r^* defined by

$$r^* C = \{x \in X \mid \forall y. xry \Rightarrow y \in C\}.$$

Unfortunately, as has been shown by Fine [4] and Thomason [22], the statement of completeness of such semantics, which is expressed as the statement:

> "for every logic L, $\vdash_L \varphi$ iff φ is valid in every Kripke frame in which every axiom of L is valid"

fails for some logics. The remedy is to extend the notion of a frame with additional information to give rise to a *modal frame*, that is, a triple $\langle X, r, T\rangle$ where $\langle X, r\rangle$ is a Kripke frame and T is a field of subsets of X (hence a Boolean algebra) closed under the operation r^*.

Definition 4.2 [19] *A* modal frame *is a triple* $F = \langle X, r, T\rangle$ *where* $\langle X, r\rangle$ *is a Kripke frame and* T *is a field of subsets of* X *closed under the operation* r^* *defined by*

$$r^* C = \{x \in X \mid \forall y . xry \Rightarrow y \in C\}.$$

The field of sets T serves a dual purpose: on the one hand, it is a modal algebra of *basic propositions* hiding inside a frame, while, on the other, it is a *clopen basis* for a topology on X. Now, enlarging the class of models in this way allows a completeness result to be proved for an arbitrary modal logic, see *e.g.* [19].

It should be noted that, given a Kripke frame $\langle X, r\rangle$, the powerset $\wp(X)$ can always serve as the field of subsets closed under r^* so every Kripke frame is a modal frame in which the basic propositions are allowed to range over *all* subsets of X, as opposed to just a *subalgebra* T. In topological terms the idea of restricting the propositions to a subalgebra corresponds to the intuition that only some of the potential propositions on X correspond to *observable* properties of the transition system, where we identify observable properties with open sets in the spirit of [21], [24], [1]. It should be noted, however, that, when we come to the extension of these ideas to the μ-calculus, we force properties which are not finitely observable (namely those expressed as fixed points) into the basis of the topology.

Observe that Kozen's original models are essentially Kripke frames, hence the interpretation of fixed points is in the complete lattice $(\wp(X), \subseteq)$, and not in the field of subsets T. This statement should not be construed as an argument for incompleteness of Kozen's system, but only as an explanation of the differences between the results presented here and that of [12].

5 Modal μ-algebras

We generalize the notion of a modal algebra to allow an indexed family of modalities. Consequently, we consider a *modal algebra* to be a pair $M = \langle B, \{\tau_a\}_{a \in \mathrm{Act}}\rangle$ where B is a Boolean algebra and $\{\tau_a\}$ is a family of operations on B satisfying $(A1)$ and $(A2)$ above. We now extend the notion of a modal algebra by adding fixed point operators as follows.

Let $M = \langle B, \{\tau_a\}\rangle$ be a modal algebra, $F = \{f_i\}_{i \in I}$ an indexed family of monotone maps $f_i : M \longrightarrow M$. M is said to *have the fixed point property for* F iff for every $f \in F$ the least fixed point of f exists. There are two important special cases of fixed point properties corresponding to the two characterizations of least fixed points given in Section 2. M is said to have the fixed point property of the *first kind* (or *second kind*) for F iff a least fixed point of the first kind (or second kind) exists for all $f \in F$.

Let $M = \langle B, \{\tau_a\}\rangle$ be a modal algebra. An *assignment* $A : V \longrightarrow M$ is a map from variables V to the elements of the modal algebra M. We define for each assignment

$A : V \longrightarrow M$ a partial function $[\![\cdot]\!]_A : L_\mu \rightarrow M$ as follows:

$$
\begin{aligned}
&[\![x]\!]_A = A(x)\\
&[\![\mathbf{tt}]\!]_A = \mathbf{tt}\\
&[\![\mathbf{ff}]\!]_A = \mathbf{ff}\\
&[\![t \wedge t']\!]_A = [\![t]\!]_A \wedge [\![t']\!]_A\\
&[\![\neg t]\!]_A = \neg [\![t]\!]_A\\
&[\![[a]t]\!]_A = \tau_a [\![t]\!]_A\\
&[\![\mu x.t]\!]_A = \mathrm{lfp}(\lambda b.[\![t]\!]_{A(x:=b)}) \quad \text{if this exists, undefined otherwise}
\end{aligned}
$$

where $\mathrm{lfp}(\lambda b.f(b))$ denotes the least fixed point of the operator $\lambda b.f(b)$ on M.

Definition 5.1 (Modal μ-algebra.) *Let M be a modal algebra. Then M is a* modal μ-algebra *iff it has the fixed point property for the family F_{L_μ} of maps given by:*

$$F_{L_\mu} = \left\{ \lambda b.[\![t]\!]_{A(x:=b)} \mid t \in L_\mu, A : V \longrightarrow M \right\}.$$

We shall say that a modal μ-algebra M is of the *first* or *second kind* if it has the fixed point property for F_{L_μ} respectively of the first or second kind. Observe that this corresponds to the existence of the meet

$$\bigwedge \left\{ b \in M \mid [\![t]\!]_{A(x:=b)} \leq b \right\}$$

for all terms t and assignments A, or the unfolding

$$(\lambda b.[\![t]\!]_{A(x:=b)})^\alpha(0)$$

for all terms t, assignments A and ordinals α.

Note that a direct consequence of Proposition 2.4 is

Proposition 5.2 *A modal μ-algebra of the second kind is also of the first kind.*

A modal algebra of the second kind in which the closure ordinals of all chains of unfoldings are less or equal a limit ordinal α is said to be *of the second α-kind*. In particular, we shall distinguish a subclass of algebras of the *second ω-kind*.

Complete modal algebras (and this includes the powerset lattices used by Kozen) are, of course, modal μ-algebras of both the first and second kind, but not necessarily of the second ω-kind (see *e.g.* the example due to Kozen [12]). In general, a modal algebra might be of one kind or the other or not of either kind. We shall show that the Lindenbaum algebra of the proof system L_μ given in Section 3 is of the first kind, though it is not known whether it is of the second. Algebras of the second kind correspond to different proof rules, in the sense that the rule (inf) above is sound as an implication in algebras of the second ω-kind (but not, in general, in algebras of the second kind or algebras of the first kind).

6 Modal μ-frames

We generalize the notion of a modal frame to allow an indexed family of accessibility relations. Thus, a *modal frame* is a triple $F = \langle X, \{r_a\}_{a \in \mathrm{Act}}, T \rangle$ where $\langle X, \{r_a\} \rangle$ is a transition system (the indexed version of a Kripke frame) and T is a field of subsets of X closed under the modality ${r_a}^*$, derived from the accessibility relation r_a, for each $a \in \mathrm{Act}$.

The field T is, at the same time, the modal algebra of basic propositions, and a clopen basis for a topology on X. In order to recover a modal algebra from a modal frame $\langle X, \{r_a\}, T\rangle$, we simply take the field T together with the operations ${r_a}^*$. More formally, any modal frame $\langle X, \{r_a\}, T\rangle$ gives rise to a modal algebra $\langle T, \{{r_a}^*\}\rangle$.

The key element in our extension of duality theory is the idea that T might also be a modal μ-algebra:

Definition 6.1 *A* modal μ-frame *is a modal frame $\langle X, \{r_a\}, T\rangle$ in which $\langle T, \{{r_a}^*\}\rangle$ is a modal μ-algebra.*

We shall say that a modal μ-frame is of the first or second kind respectively if its corresponding algebra is of the first or second kind.

There are several restrictions derived from Goldblatt [7] that are put on the structure of modal frames. These restrictions play an important role in deriving duality (and adjunction) theorems between the appropriate categories of modal algebras and frames. As they can be explained in topological terms, we shall defer their discussion to Section 9.

7 From modal μ-algebras to modal μ-frames

Let $M = \langle B, \{\tau_a\}\rangle$ be a modal μ-algebra. An established result [19] is that every modal algebra determines a modal frame. This frame is constructed by taking the set of Boolean ultrafilters $U(B)$ as the set of worlds, the relations τ_{a*} defined by:

$$S\tau_{a*}T \quad \Longleftrightarrow \quad \forall b \in B.\, \tau_a(b) \in S \Rightarrow b \in T$$

as the accessibility relations, and the family $\beta(B) = \{\beta(b) \mid b \in B\}$ of sets of ultrafilters as the basic propositions:

$$\beta(b) = \{U \text{ an ultrafilter on B} \mid b \in U\}.$$

Formally, the triple $\langle U(B), \{\tau_{a*}\}, \beta(B)\rangle$ is a modal frame, indeed the dual frame of M when we consider M purely as a modal algebra. We now propose to commence the programme of extending duality theory to the μ-construction by showing that $\langle U(B), \{\tau_{a*}\}, \beta(B)\rangle$ is a modal μ-frame. In other words, we need to show that $\langle \beta(B), \{(\tau_{a*})^*\}\rangle$ is a modal μ-algebra.

β gives rise to a mapping from assignments in M to assignments in $\langle \beta(B), \tau_{a*}\rangle$, defined by

$$\beta(A) \quad = \quad x \mapsto \beta(A(x)).$$

The following proposition shows that β commutes with the interpretation of terms.

Proposition 7.1 *For all $t \in L_\mu$, and assignments A,*

$$\beta([\![t]\!]_A) \quad = \quad [\![t]\!]_{\beta(A)}.$$

Proof: By induction on the structure of t. We know from [19] that β is an isomorphism of modal algebras, so in the base case, where t contains no fixed point operators, the result follows immediately from the definition of the interpretation function $[\![\cdot]\!]_A$.

Suppose now that the result holds for all terms t with n fixed point operators, and consider the term $\mu x.t$. If u is a fixed point of $\lambda b.[\![t]\!]_{A(x:=b)}$ then

$$\begin{aligned} \beta(u) &= \beta([\![t]\!]_{A(x:=u)}) \\ &= [\![t]\!]_{\beta(A(x:=u))} && \text{by induction hypothesis} \\ &= [\![t]\!]_{\beta A(x:=\beta(u))} \end{aligned}$$

so $\beta(u)$ is a fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$. The converse also holds because β is an isomorphism. Since β and β^{-1} form an isomorphism of modal algebras they are order preserving. Hence we have that u is the *least* fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$ if and only if $\beta(u)$ is the *least* fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$, and hence

$$\beta([\![\mu x.t]\!]_A) = [\![\mu x.t]\!]_{\beta(A)}.$$

□

We can now give the following representation theorem for modal μ-algebras extending the traditional Stone representation of Boolean algebras and the results of [7, 19].

Theorem 7.2 (Representation for modal μ-algebras) *$\langle\beta(B), \{(\tau_{a*})^*\}\rangle$ is a modal μ-algebra isomorphic to $\langle B, \{\tau_a\}\rangle$.*

Proof: Let t be any term and A be an arbitrary assignment in $\beta(B)$. We need to confirm that $[\![t]\!]_A$ is defined. Observe that A must be $\beta(A')$ for some assignment A' in M. Furthermore, $[\![t]\!]_{A'}$ must be defined since M is a modal μ-algebra. Hence

$$[\![t]\!]_A = [\![t]\!]_{\beta(A')} = \beta([\![t]\!]_{A'}) \qquad \text{Proposition 7.1}$$

Thus $[\![t]\!]_A = \beta([\![t]\!]_{A'})$ must be defined because β is a total function. So $\langle\beta(B), \{\tau_a\}\rangle$ is a modal μ-algebra, and furthermore, by Proposition 7.1, $\beta : \langle B, \{\tau_a\}\rangle \to \langle\beta(B), \{(\tau_{a*})^*\}\rangle$ is an isomorphism of modal μ-algebras. □

Furthermore, it is straightforward to prove that if $\langle\beta(B), \tau_a\rangle$ is a modal μ-algebra of the first (second) kind then so is $\langle\beta(B), \{(\tau_{a*})^*\}\rangle$.

One outcome of the reprepresentation theorem is that we have arrived at an appropriate notion of *homomorphism* between modal μ-algebras since β should be an example of such. We note that it preserves the modal structure of the algebra as well as preserving the (interpretation of) fixed points, in the sense that the image of the interpretation a fixed point is the interpretation of the fixed point in the target algebra.

8 Completeness results

In this section we derive completeness results for the systems L_μ and L_μ^{inf} with respect to certain classes of modal frames. The main difference between our results and those of Kozen is that, as we are able to obtain canonical systems, our classes of models characterise the calculi in the stronger sense, and not just for validity.

8.1 Completeness of L_μ

As a first step, we construct the Lindenbaum algebra of the proof system L_μ described in Section 3 and show that it is a modal μ-algebra of the first type. A completeness result then follows by using the representation theorem of the previous section.

Let $\simeq$ be the equivalence relation defined on propositions by $\varphi \simeq \psi$ if and only if $\vdash \varphi \iff \psi$, and let

$$\begin{aligned} [\varphi] &= \{\psi \mid \varphi \simeq \psi\} \\ L_\mu/_\simeq &= \{[\varphi] \mid \varphi \in L_\mu\}. \end{aligned}$$

We can define Boolean operations on $L_\mu/_\simeq$ in the usual way and the resulting structure $B = (L_\mu/_\simeq, \wedge, \vee, \neg, 0, 1)$ is, of course, a Boolean algebra. If we add the operators τ_a defined by

$$\tau_a[\varphi] \;=\; [\,[a]\varphi\,]$$

we have a modal algebra $\mathcal{LA}(L_\mu) = \langle B, \{\tau_a\}\rangle$, the *Lindenbaum algebra* for the modal μ-calculus.

We observe that in the Lindenbaum algebra the ordering $\leq$ on elements is defined by

$$[\varphi] \leq [\psi] \quad \text{iff} \quad \varphi \vdash \psi.$$

Lemma 8.1 *Let φ be a term with free variables $u_1, u_2, \ldots, u_n$ and $A : V \longrightarrow L_\mu$ be an assignment mapping each of the variables u_i to an equivalence class $[\theta_i]$. Then*

$$[\![\varphi]\!]_A = [\varphi(u_1 := \theta_1, \ldots, u_n := \theta_n)]$$

provided that θ_i is free for u_i in φ.

Proof: Induction on the structure of φ. $\square$

Proposition 8.2 *Let φ be a term with free variables $x, u_1, u_2, \ldots, u_n$ and $A : V \longrightarrow L_\mu$ be an assignment mapping each of the variables u_i to an equivalence class $[\theta_i]$. Then*

$$[\mu x.\varphi'] = \bigwedge \left\{[\psi] \mid [\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]\right\}$$

where $\varphi' = \varphi(u_1 := \theta_1, \ldots, u_n := \theta_n)$ provided θ_i is free for u_i in φ for all i, and ψ is free for x in φ'.

Proof: Consider any $[\psi]$ such that $[\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]$. Then, by the lemma above, we have that

$$[\varphi'(x := \psi)] = [\varphi(x := \psi, u_1 := \theta_1, \ldots, u_n := \theta_n)] \leq [\psi]$$

It follows that

$$\begin{aligned} \varphi'(x := \psi) \vdash \psi \quad &\Rightarrow \quad \mu x.\varphi' \vdash \psi \\ &\Rightarrow \quad [\mu x.\varphi'] \leq [\psi] \end{aligned}$$

Thus $\mu x.\varphi'$ is a lower bound of $\left\{[\psi] \mid [\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]\right\}$.

Furthermore

$$\begin{aligned} \varphi'(x := \mu x.\varphi) \vdash \mu x.\varphi' \quad &\Rightarrow \quad [\varphi'(x := \mu x.\varphi')] \leq [\mu x.\varphi'] \\ &\Rightarrow \quad [\mu x.\varphi'] \in \{[\psi] \mid [\varphi'(x := \psi)] \leq [\psi]\} \end{aligned}$$

Thus $[\mu x.\varphi']$ is the greatest lower bound. $\square$

Theorem 8.3 *$\mathcal{LA}(L_\mu)$ is a modal μ-algebra of the first kind.*

Proof: Immediate from the above. By α-conversion, we can always change names of bound variables so that the conditions of freeness are respected. $\Box$

Corollary 8.4 *Kozen's axiomatisation L_μ of the modal μ-calculus is complete with respect to the class of modal μ-frames of the first kind.*

Proof: As a modal μ-algebra the Lindenbaum algebra gives rise to a frame where, by our previous result, the field of sets is a modal μ-algebra of the first kind. Since the formulas valid in this frame correspond to the top element of the field of sets, the frame can serve as a canonical model for the logical system which we have described: the valid formulas must belong to the top element of the Lindenbaum algebra since the Lindenbaum algebra is isomorphic to the field of sets, and hence they must be theorems. Consequently our proof system is characterized by the canonical model, and, more generally, by the class of modal μ-frames of the first kind. $\Box$

On the positive side, our result has not only supplied a completeness result for a simple axiomatization of the calculus, but by the use of modal duality theory has exploited the notion of a canonical model in order to achieve this. On the negative side, it must be admitted that the idea of a modal μ-frame (and indeed of a modal μ-algebra) has been tailor made for just this purpose. A worthwhile future discovery, which has so far eluded the authors, would be an independent method of characterizing modal μ-frames without reference to the corresponding algebra, and a method of characterizing modal μ-algebras without reference to the interpretation of the μ-calculus.

We also emphasize that our completeness result is with respect to a different class of models than the models of [12], which are indexed Kripke frames $\langle X, \{r_a\}\rangle$, and that it does not immediately imply anything regarding completeness for Kozen's models. Every modal frame determines an indexed Kripke frame (we simply drop the basic propositions T). Every indexed Kripke frame determines a modal frame $\langle X, \{r_a\}, \wp(X)\rangle$, but this frame need not be descriptive in the sense of page 17 (for example, if X is infinite), whereas the μ-frames that arise through our representation theorem are descriptive. In other words, the space $(X, \wp(X))$ is not, in general, a Stone space. The significance of this discrepancy is that in indexed Kripke frames there is more freedom as to which sets of points may act as basic propositions (which intuitively correspond to observations); in fact, *any* set of points is allowed. Furthermore, if working with the powerset lattice, there is a cardinality mismatch between the sizes of the sets of syntactic and semantic propositions (for example, if X is countable the lattice of semantic propositions is uncountable) which does not arise in our approach.

We would also like to remark that it is not clear if Kozen's system L_μ is complete with respect the modal μ-algebras of the second kind. As will be seen in the next section, algebras of the second ω-kind correspond in a certain sense to the infinitary rule (inf).

8.2 Completeness of the infinitary system L_μ^{inf}

In this section we investigate the role of algebras of the second-ω kind and derive a completeness result for the system L_μ^{inf} described in Section 3.

The difficulty with the rule (inf), already observed in [13], is that it is *not* valid as an implication, since in the powerset lattice it is not true in general that the interpretation of $\mu x.\varphi$ is below the interpretation of ψ whenever the latter is an upper bound of all finite unfoldings. However, if it is the case that the interpretation of ψ is above all finite unfoldings of φ *in all models* then the interpretation of $\mu x.\varphi$ is below ψ in all models. The inductive rule due to Walukiewicz [23] suffers from a similar problem.

On the other hand, the rule (inf) is sound as an implication in algebras of the second ω-kind. This is because if the closure ordinal is ω then the interpretation of $\mu x.\varphi$ is equal to the ω-th unfolding of φ, which is by definition the ω-join of φ^i for $i \in \omega$. Hence, if for all $i \in \omega$ we have $\varphi^i \leq \psi$ it follows that $\mu x.\varphi \leq \psi$.

Let us now consider the Lindenbaum algebra $\mathcal{LA}(L_\mu^{\mathrm{inf}})$ of the system L_μ^{inf} build in the usual way. The following proposition provides an alternative characterization of the least fixed point formulas as the ω-joins of finite unfoldings.

Proposition 8.5 *Let $\varphi \in L_\mu$ be a term. Then*

$$[\mu x.\varphi] = \bigvee \left\{[\varphi^i] \mid i \in \omega\right\}.$$

Proof: We first show that $[\mu x.\varphi]$ is an upper bound for $\left\{[\varphi^i] \mid i \in \omega\right\}$ (proof by induction on φ^i). Clearly $[\varphi^0] \leq [\mu x.\varphi]$. For the induction step suppose $[\varphi^i] \leq [\mu x.\varphi]$. By monotonicity of φ

$$\begin{array}{lll} \varphi^i \vdash \mu x.\varphi & \Rightarrow & \varphi(x := \varphi^i) \vdash \varphi(x := \mu x.\varphi) \\ & \Rightarrow & \varphi^{i+1} \vdash \mu x.\varphi \qquad\qquad\qquad \text{by } (\mu\mathrm{R}) \\ & \Rightarrow & [\varphi^{i+1}] \leq [\mu x.\varphi] \end{array}$$

It remains to show that $[\mu x.\varphi]$ is the least upper bound of $\left\{[\varphi^i] \mid i \in \omega\right\}$. We have

$$\begin{array}{lll} \forall i \in \omega.[\varphi^i] \leq [\psi] & \Rightarrow & \forall i \in \omega.\varphi^i \vdash \psi \\ & \Rightarrow & \mu x.\varphi \vdash \psi \qquad \text{by (inf)} \\ & \Rightarrow & [\mu x.\varphi] \leq [\psi] \end{array}$$

□

Theorem 8.6 $\mathcal{LA}(L_\mu^{\mathrm{inf}})$ *is a modal μ-algebra of the second ω-kind.*

Proof: Immediate from the above. □

Corollary 8.7 *Kozen's axiomatisation L_μ^{inf} of the modal μ-calculus is complete with respect to the class of modal μ-frames which are of the second ω-kind.*

Proof: Similar to the proof of Corollary 8.4. □

Since all algebras of the second kind are also of the first kind, the importance of the class of models for which we have obtained the above completeness statement is in that in these models the two interpretations of fixed points (as the minimum of the set of pre-fixed-points and as the ω-join of finite unfoldings) coincide. This fact is potentially useful in proofs of results concerning the modal μ-calculus.

The system L_μ^{inf} is known to be complete for validity with respect to the indexed Kripke frames [13]. These Kripke frames $\langle X, \{r_a\}\rangle$ give rise to modal frames $\langle X, \{r_a\}, \wp(X)\rangle$, which are frames of the second kind, but *not*, in general, of the second ω-kind. Thus, we have obtained a completeness result for a different class of models, namely those triples

$\langle X, \{r_a\}, T\rangle$ which of the second ω-kind. The infinitary system L_μ^{inf} is axiomatically characterised by our class in the stronger sense.

Furthermore, since we have obtained canonical systems, we can deduce that our systems are compact in the logical sense [9]. This need not be the case for non-canonical systems.

9 Duality theory

In this section we derive a duality theory for modal μ-algebras. We consider the special cases of modal μ-algebras of the first and second kind separately. We complete the section by deriving a duality theorem for algebras of the second ω-kind.

9.1 Duality for modal algebras

We begin by reviewing the existing duality theory for modal algebras. This duality is based on the original Stone duality (see Section 1) for Boolean algebras, since modal algebras $\langle B, \{\tau_a\}_{a\in \mathrm{Act}}\rangle$ are Boolean algebras B with an indexed family of operators $\{\tau_a\}$ on B. Homomorphism of modal algebras are Boolean algebra morphisms which, in addition, preserve each τ_a.

Following Halmos [8] and Sambin and Vaccaro [19], we think of the category Mal of modal algebras and homomorphisms as obtained from a larger auxiliary category Bal of Boolean algebras and hemimorphisms (*i.e.* maps $\tau : B \longrightarrow B'$ preserving finite meets). Strictly speaking when modalities are indexed by actions, the categories Mal and Bal are also parameterized on the set of actions. However, for notational simplicty we omit the index, since the categories we have in mind have been fixed by our choice of Act. A modal algebra $\langle B, \{\tau_a\}\rangle$ (an *object* in Mal) can be identified with an indexed family of arrows $B \stackrel{\tau_a}{\to} B$ in Bal, and the *arrows* of Mal are those morphisms h in Bal which make each of the following squares commute:

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "\tau_a"'] & B' \arrow[d, "\tau'_a"] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

The category Fra, fixed by our choice of Act, has as objects (indexed) modal frames $\langle X, \{r_a\}_{a\in\mathrm{Act}}, T\rangle$ and as arrows *weak contractions* c, that is, maps which are continuous in the topology on X induced by T satisfying $\mathrm{Cl}(r_a c x) = \mathrm{Cl}(c r_a x)$ for all $a \in \mathrm{Act}$, $x \in X'$, where $\mathrm{Cl}(x)$ denotes the closure of x. We would like to reproduce the view of modal algebras as "living" in a larger category for frames as well. Modal frames $\langle X, \{r_a\}, T\rangle$ can be viewed as indexed families of arrows $\langle X, T\rangle \stackrel{r_a}{\to} \langle X, T\rangle$ in the (larger) category Spa of *spaces* $\langle X, T\rangle$ (where X denotes the set of points and T is a field of subsets of X serving as a clopen basis of the topology) with *continuous relations* in the sense defined below as morphisms.

Definition 9.1 *Let $\langle X, T\rangle$ and $\langle Y, U\rangle$ be spaces. $r \subseteq X \times Y$ is a continuous relation iff, for every $D \in U$, $r^*D \in T$ where*

$$r^*D \;=\; \{x \in X \mid \forall y \in Y. xry \Rightarrow y \in D\}.$$

The phrase "continuous relation" is derived from [19]. This notion bears some resemblance to the idea of an *upper semi-continuous multifunction* as found in, for example [21]. However, there the "inverse image" of an open is an open, whereas here the "inverse image" of a clopen in the basis is a clopen in the basis. Note, however, that when r is a point-closed relation (see (F1) below) between Stone spaces then the two notions coincide.

Thus, arrows on frames are those weak contractions c which make each of the following diagrams commute:

$$\begin{array}{ccc}
(X,T) & \xleftarrow{c} & (X',T') \\
\uparrow r_a & & \uparrow r'_a \\
(X,T) & \xleftarrow[c]{} & (X',T')
\end{array}$$

There are functors $(\cdot)_*$ from Mal to Fra and $(\cdot)^*$ from Fra to Mal whose action on objects is given by:

$$\begin{aligned}
(\langle B, \{\tau_a\}\rangle)_* &= (\langle U(B), \tau_{a*}, \beta(B)\rangle) \\
(\langle X, \{r_a\}, T\rangle)^* &= (\langle T, r_a^*\rangle)
\end{aligned}$$

These functors give rise to an adjunction, which restricts to a duality for the subcategory DFra of descriptive frames. A modal frame $F = \langle X, \{r_a\}, T\rangle$ is said to be *refined* iff the following conditions hold:

$$\begin{array}{lll}
\text{(F1)} & \forall a \in \text{Act}.(x r_a y \iff \forall C \in T. x \in {r_a}^* C \Rightarrow y \in C) & (r_a \text{ is point-closed}) \\
\text{(F2)} & \forall C \in T.(x \in C \iff y \in C) \Rightarrow x = y & ((X,T) \text{ is Hausdorff})
\end{array}$$

In addition, if F satisfies:

$$\text{(F3)} \quad \forall V \in \wp(T), V \text{ an ultrafilter}, \bigcap_{S \in V} S = \{x\} \text{ for some } x \in X \quad ((X,T) \text{ is compact})$$

then it is called *descriptive* after Goldblatt [7]. Axiom (F1) guarantees that the modalities can be recovered from the relations r_a. Axiom (F2) can be explained as the Hausdorff separation axiom, since its meaning is that any two points with the same system of neighbourhoods are equal. Finally, axiom (F3) enforces compactness, since it states that every ultrafilter of neighbourhoods converges, and hence must have a non-empty intersection (which must be the singleton set $\{x\}$ by (F2)).

DFra is the category of descriptive frames with *contractions* (*i.e.* maps

$$c : \langle X', \{r'_a\}, T'\rangle \longrightarrow \langle X, \{r_a\}, T\rangle$$

such that c is continuous and satisfies the stronger condition $cr'_a x = r_a cx$ for every $a \in \text{Act}$, $x \in X'$).

Theorem 9.2 [19] (Modal duality) *The categories* Mal *of modal algebras and homomorphisms, and* DFra *of descriptive frames and contractions, are dual to each other.*

Our intention is to extend the above duality theorem to the case of modal μ-algebras.

9.2 Duality for Boolean algebras with infinite operations

Recall that we have identified two possible interpretations of a (least) fixed point term: a (possibly infinite) meet of pre-fixed-points, or a (possibly infinite) join of its unfoldings. Thus, we may be led to believe that if we can characterize the maps on frames which induce modal homomorphisms preserving infinite joins (and meets), then we can show that these homomorphisms also preserve fixed points. As we shall see in later sections, this is not quite the case.

We now overview some standard results concerning Boolean algebras with some infinite joins found in [20] in the hope of using the characterization of morphisms to derive our duality theorems.

Let B, B' be Boolean algebras, and let $h : B \longrightarrow B'$ be a Boolean algebra homomorphism. h is said to *preserve the join* $\bigvee_{i \in I} a_i$, where I is an index set, if, and only if,

$$h(\bigvee_{i \in I} a_i) \quad = \quad \bigvee_{i \in I} h(a_i).$$

Let β be the isomorphism of a Boolean algebra B onto the clopen subsets of the Stone space $(U(B), \beta(B))$ of B, and let $a = \bigvee_{i \in I} a_i$. Since β is an isomorphism, the join of $\beta(a_i)$ must exist in the algebra of clopens, but it does not have to be the set-theoretic union. It can be shown [20] that for a Boolean algebra B the joins, if they exist, are the (topological) *closures* of certain unions in the Stone space of B, in the sense stated below.

Proposition 9.3 *Let β be the isomorphism of a Boolean algebra B onto the clopen subsets of the Stone space of B. For any element $a = \bigvee_{i \in I} a_i$, where I is an index set,*

$$a \quad = \quad \bigvee_{i \in I} a_i$$

iff

$$\beta(a) = \mathrm{Cl}(\bigcup_{i \in I} \beta(a_i)).$$

iff the defect *set*

$$S \quad = \quad \beta(a) \setminus \bigcup_{i \in I} \beta(a_i)$$

is nowhere dense, i.e. $\mathrm{Int}(\mathrm{Cl}(S)) = \emptyset$.

The elements of the set S are all and only those ultrafilters which contain $\bigvee_{i \in I} a_i$ without containing any of the a_i. Such an ultrafilter, often called *non-standard* (non-principal), is not κ-complete (where κ is the cardinality of I). So, if an infinite join exists, then the set-theoretic difference between its β-image and the union of the β-images of its approximants is nowhere dense, and cannot contain a β-image of an element of the Boolean algebra. We call such sets *defect sets* following Sikorski [20]. We can regard the defect sets as negligible, since, for example, in measure theory nowhere dense sets are of measure 0. A similar point about non-standard ultrafilters and nowhere dense sets was also made in [11] for the case of PDL.

Observe that, even if the Boolean algebra B in question is complete, its β-image is not necessarily a complete field of sets, that is, a field of sets closed under all unions (and hence intersections) of sets.

The case of infinite meets is dual.

We remark that, traditionally, Proposition 9.3 has been used to formulate a duality theorem between the category of κ-complete Boolean algebras with homorphisms preserving κ-joins (and hence κ-meets) and the category of Stone spaces and *κ-continuous* functions (*i.e.* functions which are continuous and, in addition, preserve nowhere-dense subsets of κ-closed sets under inverse image, where a set is defined to be *κ-closed* iff it is an intersection of κ clopens). Since continuous functions preserve closed sets under inverse image, the property of being κ-closed is preserved, but the property of being nowhere dense need not be so, and hence the additional condition is required.

We shall use Proposition 9.3 to formulate a duality theorem for the more general setting of modal μ-algebras (of the second kind).

9.3 Duality for modal μ-algebras

Let M be a modal μ-algebra. We have already remarked that a suitable notion of a homomorphism for modal μ-algebras must be a modal algebra homomorphism which, in addition, preserves the interpretation of fixed point terms, in the sense that the image of the interpretation of a fixed point term is the interpretation of the fixed point in the target algebra. We now give the formal definition.

Definition 9.4 (Modal μ-homomorphism) *Let M, M' be modal μ-algebras. $h : M \longrightarrow M'$ is a* modal μ-homomorphism *iff it is a homomorphism of modal algebras and*

$$h([\![\mu x.t]\!]_A) = [\![\mu x.t]\!]_{h(A)}$$

for all terms t in M and assignments $A : V \longrightarrow M$.

Since we have a representation for modal μ-algebras in terms of modal μ-frames, it follows that the category $\mu\mathsf{Mal}$ of modal μ-algebras and homomorphisms has a dual, which is a category of modal μ-frames and contractions between them, but we do not have an elementary characterization of the maps as yet. The same applies to the case of the category $\mu\mathsf{Mal}^{\mathbf{1}}$ of modal μ-algebras of the first kind. It should be noted that even if the morphisms preserve infinite meets, it does *not* follow that they preserve the (least) fixed points, since the meet of the set of the pre-fixed-points is mapped *into* the set of pre-fixed-points, but it does not follow that the minimum of the image of pre-fixed-points coincides with the minimum of the set of pre-fixed-points in the target algebra. (For example, there can be a pre-fixed-point in the target algebra, which is *below* the homomorphic image of the pre-fixed-points.)

Fortunately, we are in a better position with the second kind of algebras, as can be seen from the following statement. This is at a cost of requiring that modal μ-homomorphisms satisfy the additional condition of being *smooth*.

Definition 9.5 (Smooth μ-homomorphism) *Let M, M' be modal μ-algebras of the second kind, and let $h : M \longrightarrow M'$ be a modal μ-homomorphism. h is a* smooth *homorphism iff for every limit ordinal λ, term t and assignment A in M, h preserves the join of all the α unfoldings of the operator $\lambda b.[\![t]\!]_{A(x:=b)}$, for $\alpha < \lambda$. Formally,*

$$h(\bigvee_{\alpha<\lambda} (\lambda b.[\![t]\!]_{A(x:=b)})^{\alpha}(0)) \quad = \quad \bigvee_{\alpha<\lambda} (\lambda b.[\![t]\!]_{hA(x:=b)})^{\alpha}(0).$$

The intent of the above definition is to prevent the homomorphisms from mapping the interpretation of a fixed point term to the interpretation of a fixed point in the target algebra *without* preserving the joins in the chains of unfoldings approximating the fixed points. (For example, we may have a homomorphism which maps the ω unfolding to the $\omega+1$ unfolding but nevertheless preserves the least fixed point obtained at some ordinal $\alpha > \omega$.)

We know from Sikorski (see Proposition 9.3) that a homorphism preserving an infinite join is induced by a continuous function mapping the corresponding defect set to a nowhere dense set under inverse image. This allows us to derive the following characterization of smooth modal μ-homomorphisms.

Proposition 9.6 *Let M, M' be modal μ-algebras of the second kind, and let $h : M \longrightarrow M'$ be a modal μ-homomorphism. Then h is a smooth μ-homomorphism iff its dual h_* maps each of the following defect sets to a nowhere dense set under inverse image.*

Defect sets are all

$$\mathrm{Cl}(\bigcup S_{t,A,\alpha}) \setminus \bigcup S_{t,A,\alpha}$$

where $S_{t,A,\alpha} = \left\{ (\lambda b.[\![t]\!]_{A(x:=b)})^{\beta} \mid \beta < \alpha \right\}$ *for some term t, assignment A in M and ordinal α.*

The following duality statement now follows.

Theorem 9.7 (Duality for modal μ-algebras of the second kind) *The category $\mu\mathsf{Mal}^{\mathbf{2}}$ of the modal μ-algebras of the second kind with smooth homomorphisms is dual to the category $\mu\mathsf{DFra}^{\mathbf{2}}$ of descriptive modal μ-frames of the second kind and contractions mapping subsets of defect sets to nowhere dense sets under inverse image.*

Recall that we have so far provided an axiomatic characterization for algebras of the first kind, but no duality statement with an elementary characterization of the morphims. For algebras of the second kind we produced a full duality statement (restricting the morphisms to the smooth homomorphisms), but no axiomatic characterization as such. We can do better for algebras which are simultaneously of the first and second ω-kind, which are characterised by the system L_{μ}^{inf}.

Let ${\mu\mathsf{Mal}^{\mathbf{2}}}_{\omega}$ denote the full subcategory of $\mu\mathsf{Mal}^{\mathbf{2}}$ whose objects are of the second ω-kind and homorphisms between them (note that *all* homomorphisms in ${\mu\mathsf{Mal}^{\mathbf{2}}}_{\omega}$ are smooth). The following duality statement now follows.

Theorem 9.8 (Duality for modal μ-algebras of the second ω-kind) *The category ${\mu\mathsf{Mal}^{\mathbf{2}}}_{\omega}$ with homomorphisms is dual to the category $\mu\mathsf{DFra}_{\omega}$ of descriptive modal μ-frames which are of the second ω-kind and contractions mapping subsets of defect sets to nowhere dense sets under inverse image.*

10 Further work

Despite the positive aspect of these results, there is still room for doubt as to whether the canonical model, and hence the class of modal μ-frames, has quite the properties which we want in a class of models for a sensible axiomatization of the μ-calculus. The problem is caused by the existence of so-called *non-standard* ultrafilters. Consider the following set of sentences.

$$\{\neg\langle a\rangle^{n}\varphi \mid n \in \omega\} \cup \{\mu x.\varphi \vee [a]x\}$$

The formulas on the left express the property that for each $n \in \omega$, it is not possible to execute n actions a and get to a state where φ holds. The formula on the right expresses the property that eventually, by repeating a actions, we can get into a state where φ holds. At first sight these properties are a contradiction. However, the set S is consistent according to the system of rules that we have supplied. It follows that the corresponding set of elements in the Lindenbaum algebra has the finite intersection property and so can be extended to an ultrafilter. Thus, there exists a world in the canonical model which satisfies the set S.

Such a world is provided by the following 'Zeno' machine which from state s_0 performs transfinitely many a actions to eventually get to a state t_0 where φ holds.

$$s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3 \longrightarrow . \quad . \; . \, . . . \; . . . \, . \; . \quad . \longrightarrow t_3 \longrightarrow t_2 \longrightarrow t_1 \longrightarrow t_0$$

The topology is generated by the basis of clopens containing all finite subsets of $\{t_n \mid n \in \omega\}$. None of the states t_i is actually accessible from a state s_j. The link between them is specified in the topology: there is no open set which contains all of the s_j without containing all of the t_i as well. This space is certainly compact. If one were to identify all of the states s_j then it would become totally separated as well and hence a Stone space. Since we have allowed non-standard ultrafilters, this kind of situation must actually occur in the canonical model.

Thus we see that the non-standard ultrafilters are an indispensable part of the canonical model for the proof system described in this paper. The question remains whether a sensible version of the μ-calculus would allow such ultrafilters as possible worlds. It is relatively easy to show that frames which omit such ultrafilters give rise to a non-compact topology when the field of sets is used as a basis (in contrast to our canonical frame, which gives rise to a Stone space). Further work is needed to establish whether such non-compact μ-frames characterize an acceptable proof system. We anticipate that this may involve proof systems with ω- or induction rules. If one were to rule out non-standard ultrafilters, then the canonical model could be obtained via a representation of, *e.g.*, σ-algebras as σ-fields of sets. We would like to note that this alternative approach would take us outside the traditional Stone duality because of non-compactness.

11 Conclusion

We have obtained completeness results for Kozen's systems L_μ and L_μ^{inf} (but with respect to a different class of models from that discussed by Kozen). We have introduced the notion of a modal μ-algebra, in which the interpretation of fixed point terms is required to exist. We have identified two basic kinds of modal μ-algebras, corresponding to two different formulations of (least) fixed points (as minima of sets of pre-fixed-points and joins of chains of unfoldings), and explained their interplay with the proof rules. Finally, we have also shown that certain classes of modal μ-algebras admit natural duality theorems.

It is interesting that a recent paper by Bonsangue *et al* [2] proposed an alternative duality theory motivated by fairness issues. The main difference lies in the way that infinite joins (which arise from fixed points) are interpreted: our approach interprets them as the *joins* in the subalgebra of clopens (in other words, the *closures* of the unions), whereas in

[2] they are interpreted as the *unions*. One positive aspect of the latter approach is that it adequately models infinite logical disjunction (elements are in the join iff they are in one of the disjuncts). However, one must work outside the traditional Stone duality, with an infinitary logic. In contrast, our approach is within compact spaces, thus admitting a finitary logical characterization, and hence model checking. We note that the infinite join may contain points which are not in any of the approximants, which may account for an unnatural interpretation of infinite disjunction. These added points, however, are precisely the so called non-standard (non-principal) ultrafilters, which should be thought of as idealised objects, or limits, that do not exist in reality, but are needed to enforce the finitariness restrictions in the theory.

The work is so far independent of Abramsky's "domain theory in logical form", which is based on the more general Stone representation of distributive lattice duality. We anticipate that an extension of the results presented here to the negation-free fragment of the modal μ-calculus would be straightforward, and this would enable us to draw some connections with Abramsky's work.

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