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Domain Equations for Probabilistic Processes

CHRISTEL BAIER[1] and MARTA KWIATKOWSKA[2†]

[1] *Fakultät für Mathematik & Informatik, Universität Mannheim, 68131 Mannheim, Germany.* baier@pi2.informatik.uni-mannheim.de

[2] *School of Computer Science, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK.* M.Z.Kwiatkowska@cs.bham.ac.uk

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In this paper we consider Milner's calculus *CCS* enriched by a probabilistic choice operator. The calculus is given operational semantics based on probabilistic transition systems. We define operational notions of preorder and equivalence as probabilistic extensions of the simulation preorder and the bisimulation equivalence respectively. We extend existing category-theoretic techniques for solving domain equations to the probabilistic case and give two denotational semantics for the calculus. The first, "smooth", semantic model arises as a solution of a domain equation involving the probabilistic powerdomain and solved in the category $CONT_\perp$ of continuous domains. The second model also involves appropriately restricted probabilistic powerdomain, but is constructed in the category *CUM* of complete ultra-metric spaces, and hence is necessarily "discrete". We show that the domain-theoretic semantics is fully abstract with respect to the simulation preorder, and that the metric semantics is fully abstract with respect to bisimulation.

Contents

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1. Introduction

To deal with situations involving uncertainty, which arise, for example, in distributed systems, fault-tolerant systems, communication protocols, it has become necessary to extend existing techniques to the probabilistic case. Concurrent calculi such as *CCS* (Milner 1989) and *CSP* (Hoare 1985) can successfully serve as high-level specification languages for compositional design and analysis of distributed systems; their theoretical foundations, i.e. the underlying transition systems, associated operational notions and logics, as well as the denotational models, is considered to be well understood, which makes them eminently suitable to serve as basis for formulating probabilistic calculi.

In this paper we consider the (full) calculus of *CCS* (Milner 1989), which we enrich with the probabilistic choice operator, instead of probabilistic choice replacing the usual non-deterministic choice. Non-determinism is used to model situations where two possibilities arise, and it is not known which would be taken. While genuinely useful in cases such as underspecification, possibly to be removed by further refinements, non-deterministic choice is not amenable when *quantitative* analysis is needed, for example, when the (probabilistic) distribution of possible continuations is known. By means of the probabilistic choice we add to *CCS* it is possible to state that the process, having executed an action, becomes another process *with probability* λ. Our framework allows to independently combine both the quantitative and qualitative forms of choice. This means that it is possible in a given state to non-deterministically choose between two or more probabilistic distributions on the successor states. The distinction between non-deterministic and probabilistic choices is necessary since concurrent systems contain states that are inherently non-deterministic. The non-determinism may arise e.g. from the asynchronicity of certain subprocesses. It is conventioned that a scheduler decides which of the subprocesses performs its next step, and that no (probabilistic) assumptions about these decisions, which any scheduler has to obey, are or can be made.

Research on probabilistic concurrent calculi has so far concentrated mainly on the operational notions and the associated logics, see e.g. (Larsen and Skou 1991; Segala and Lynch 1994; van Glabbeek *et al.* 1990; Jou and Smolka 1990; Christoff 1990; Giacalone *et al.* 1990; Hansson and Jonsson 1990; Lowe 1991; Tofts 1994; Cleaveland *et al.* 1992; Yuen *et al.* 1994; Yi 1994; Yi and Larsen 1992). Larsen & Skou (Larsen and Skou 1991) defined probabilistic transition systems and probabilistic bisimulation. Segala & Lynch (Segala and Lynch 1994) introduced the notion of probabilistic simulation. Probabilistic extensions of *CCS* are introduced in (Giacalone *et al.* 1990; Hansson and Jonsson 1990; Tofts 1994; Yi and Larsen 1992; Yi 1994) and Probabilistic *CSP* in (Lowe 1991). While the calculi in (Giacalone *et al.* 1990; Tofts 1994) are based on *SCCS* (Milner 1983) and replace the non-deterministic choice operator by a probabilistic choice operator, similarly to (Yi and Larsen 1992; Yi 1994) we deal with a calculus that is based on (asynchronous) *CCS* and allows for both non-deterministic and probabilistic choice. A "metric" for ϵ-bisimulation is introduced in (Giacalone *et al.* 1990), but, unlike our approach, does not satisfy the axioms of a metric.

In this paper we focus on extending to the probabilistic setting existing category-theoretic techniques for providing a (non-probabilistic) calculus with fully abstract denotational semantics. Denotational semantics, being compositional, provides the theory that underpins system decomposition; in addition, if fully abstract, i.e., if the inherent order or equality in the model precisely corresponds to the operational (pre)order or equivalence, the denotational semantics can provide additional insight into the nature of operational notions, and eventually serve as an intermediate link between the operational semantics and an appropriate logic.

In contrast to the fully abstract models for probabilistic processes, which are based on testing equivalences, see e.g. (Jonsson and Yi 1995), we take as basis Probabilistic *CCS* together with the probabilistic bisimulation equivalence (Larsen and Skou 1991) and the probabilistic simulation preorder (Segala and Lynch 1994). We include both the domain-theoretic and the metric-space approach in our investigations. Starting from domain equations for (non-probabilistic) synchronization trees, we derive domain equations involving the probabilistic powerdomain construction which, when solved respectively in the categories $CONT_\perp$ of continuous domains and *CUM* of complete ultra-metric spaces, give rise to semantic "domains" of probabilistic trees. We are able to show that the inherent order on the domain of processes corresponds to the probabilistic simulation, while the equality on the metric space to the probabilistic bisimulation.

The paper is organised as follows. Section 2 introduces probabilistic transition systems, together with probabilistic simulation and bisimulation. In Sections 3 and 4 we give domain equations for probabilistic trees and the denotational models they give rise to, with proofs relegated to Appendix. Finally, in Section 5 we discuss operational and denotational semantics for Probabilistic *CCS* and state the corresponding full abstraction results. Section 6 concludes the paper.

Preliminary version of this paper appeared as (Baier and Kwiatkowska 1997).

2. Probabilistic transition systems

In this section we give a brief overview of the notions of probabilistic distributions (Section 2.1), probabilistic transition systems (Section 2.2), and the probabilistic bisimulation and simulation (Section 2.3). Our model of probabilistic transition systems is based on the *simple probabilistic automaton* of (Segala and Lynch 1994). It generalizes the *concurrent Markov chains* considered e.g. in (Courcoubetis and Yannakakis 1995; Vardi 1985) and the *alternating model* of (Hansson and Jonsson 1990).

2.1. *Distributions*

In this section we introduce our notation concerning probability distributions. We only consider discrete distributions, which are sufficient for our purposes since we use them to model processes with a probabilistic choice operator which specifies the probabilities of the possible outcomes over a set of countably many alternatives. This differs from e.g. the approach of (Blute *et al.* 1997), where continuous-space processes (with possibly uncountably many alternatives of probabilistic choices) are considered.

Let X be a set. We write $\mathcal{D}_1(X)$ to denote the set of all (probabilistic) *distributions* on X, i.e. the set of all functions $\mu : X \to [0,1]$ such that $\mu(x) \neq 0$ for at most countably many $x \in X$ and

$$\sum_{x \in X} \mu(x) \;=\; 1.$$

A distribution μ is called *simple* iff there exists $z \in X$ with $\mu(z) = 1$ (and $\mu(x) = 0$ for all $x \neq z$). For $z \in X$, μ_z^1 denotes the unique simple distribution on X with $\mu_z^1(z) = 1$. We extend a distribution μ to a function $2^X \to [0,1]$, $U \mapsto \mu[U]$, which assigns to each subset U of X the probability

$$\mu[U] \;\stackrel{\text{def}}{=}\; \sum_{x \in U} \mu(x).$$

We often use the composition of distributions. Let X, Y be sets and μ, ν distributions on X and Y respectively. The *composition* $\mu * \nu$ is a distribution on $X \times Y$:

$$\mu * \nu : X \times Y \to [0,1], \quad (\mu * \nu)(x,y) \;\stackrel{\text{def}}{=}\; \mu(x) \cdot \nu(y).$$

Definition 2.1. Let X, Y be sets and $R \subseteq X \times Y$ and $\mu \in \mathcal{D}_1(X)$, $\nu \in \mathcal{D}_1(Y)$. A *weight function* for (μ, ν) w.r.t. R is a function $\delta : X \times Y \to [0,1]$ which satisfies:

1. $\delta(x,y) \neq 0$ for at most countably many $(x,y) \in X \times Y$.
2. For all $x \in X$, $y \in Y$:
$$\sum_{y \in Y} \delta(x,y) \;=\; \mu(x), \qquad \sum_{x \in X} \delta(x,y) \;=\; \nu(y)$$
3. If $\delta(x,y) > 0$ then $(x,y) \in R$.

We write $\mu \preceq_R \nu$ if there exists a weight function for (μ, ν) w.r.t. R.

Clearly, if $R_1 \subseteq R_2$ then each weight function w.r.t. R_1 is also a weight function w.r.t. R_2. Hence, $\mu \preceq_{R_1} \nu$ implies $\mu \preceq_{R_2} \nu$.

Remark 2.2. Let X, Y, Z be sets, μ_X, μ_Y and μ_Z distributions on X, Y, Z respectively, and let $R_{X,Y} \subseteq X \times Y$, $R_{Y,Z} \subseteq Y \times Z$ and $\delta_{X,Y} : X \times Y \to [0,1]$, $\delta_{Y,Z} : Y \times Z \to [0,1]$

weight functions for (μ_X, μ_Y) w.r.t. $R_{X,Y}$, resp. for (μ_Y, μ_Z) w.r.t. $R_{Y,Z}$. Then,

$$\delta_{X,Z} : X \times Z \to [0,1], \quad \delta_{X,Z}(x,z) \stackrel{\text{def}}{=} \sum_{y \in Y_0} \frac{\delta_{X,Y}(x,y) \cdot \delta_{Y,Z}(y,z)}{\mu_Y(y)}$$

is a weight function for (μ_X, μ_Z) w.r.t.

$$R_{X,Y} \circ R_{Y,Z} \stackrel{\text{def}}{=} \{ (x,z) \in X \times Z : \exists\, y \in Y \ ((x,y) \in R_{X,Y} \wedge (y,z) \in R_{Y,Z}) \}.$$

In the above, Y_0 denotes the set of $y \in Y$ with $\mu_Y(y) \neq 0$. Thus, if $\mu_X \preceq_{R_{X,Y}} \mu_Y$ and $\mu_Y \preceq_{R_{Y,Z}} \mu_Z$ then $\mu_X \preceq_R \mu_Z$ where $R = R_{X,Y} \circ R_{Y,Z}$. In particular, if X is a set and $R \subseteq X \times X$ is a transitive relation then $\preceq_R$ is a transitive relation on $\mathcal{D}_1(X)$. From this, if R is a preorder on X (i.e. R is reflexive and transitive) then $\preceq_R$ is a preorder on $\mathcal{D}_1(X)$. ■

Remark 2.3. Let X, Y be sets, $R \subseteq X \times Y$, $\mu \in \mathcal{D}_1(X)$, $\mu' \in \mathcal{D}_1(Y)$ such that $\mu \preceq_R \mu'$. Then, $\mu' \preceq_{R^{-1}} \mu$ where $R^{-1} \stackrel{\text{def}}{=} \{(y,x) : (x,y) \in R\}$. ■

If $f : X \to Y$ is a function then $\mathcal{D}_1(f) : \mathcal{D}_1(X) \to \mathcal{D}_1(Y)$ is given by

$$\mathcal{D}_1(f)(\mu)(y) \stackrel{\text{def}}{=} \mu[\, f^{-1}(y)\,].$$

Hence, $\mathcal{D}_1$ can be considered as endofunctor of *SET*, the category of sets and functions.

Remark 2.4. Let $f : X \to Y$ be a function. Then,

$$\delta : X \times Y \to [0,1], \quad \delta(x,y) \stackrel{\text{def}}{=} \begin{cases} \mu(x) & : \text{ if } f(x) = y \\ 0 & : \text{ otherwise} \end{cases}$$

is a weight function for $(\mu, \mathcal{D}_1(f)(\mu))$ w.r.t. $R \stackrel{\text{def}}{=} \{(x, f(x)) : x \in X\}$. Thus, $\mu \preceq_R \mathcal{D}_1(f)(\mu)$. ■

2.2. *Probabilistic transition systems*

In what follows, we suppose *Act* to be a nonempty and countable set of actions. A *probabilistic transition system* is a tuple $(S, \to)$ where S is a set of states and $\to$ a transition relation

$$\to \ \subseteq \ S \times Act \times \mathcal{D}_1(S).$$

We write $s \xrightarrow{\alpha} \mu$ instead of $(s, \alpha, \mu) \in \to$. $(S, \to)$ is called

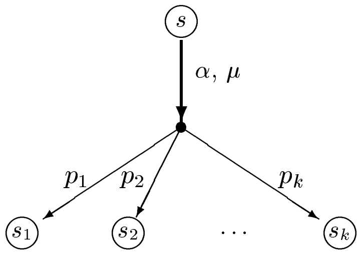
- *finitely branching* iff for each $s \in S$ the set $\{(\alpha, \mu) : s \xrightarrow{\alpha} \mu\}$ is finite;
- *reactive* iff for each $\alpha \in Act$ there is at most one transition $s \xrightarrow{\alpha} \mu$, cf. (van Glabbeek *et al.* 1990);
- *image-finite* iff for all $s \in S$, $\alpha \in Act$ the set $\{\mu \in \mathcal{D}_1(S) : s \xrightarrow{\alpha} \mu\}$ is finite.

We define *act*(s) to be the set of actions $\alpha \in Act$ such that there exists a transition $s \xrightarrow{\alpha} \mu$. A state s is called *terminal* iff *act*$(s) = \emptyset$, i.e. iff there is no transition starting in s. The underlying graph of $(S, \to)$ is the directed graph whose nodes are the states $s \in S$ and which contains an edge from s to t if and only if there exists a transition $s \xrightarrow{\alpha} \mu$ with

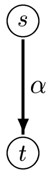
$\mu(t) > 0$. We say that a state t is *reachable* from a state s iff there exists a path from s to t in the underlying graph.

Non-probabilistic transition systems (where the transition relation $\rightarrow$ is a subset of $S \times Act \times S$) arise as special cases of probabilistic transition systems by identifying each "non-probabilistic" transition $s \xrightarrow{\alpha} t$ with the "probabilistic" transition $s \xrightarrow{\alpha} \mu_t^1$.

We draw probabilistic transition system as follows. The states are in circles. A transition $s \xrightarrow{\alpha} \mu$ where μ is not simple is depicted as:



where $\mu(s_i) = p_i > 0$ and $\sum_i p_i = 1$. The names of the distributions are omitted if they are not needed. Transitions of the form $s \xrightarrow{\alpha} \mu_t^1$ are depicted as:



2.3. *Probabilistic bisimulation and simulation*

Following (Larsen and Skou 1991; Segala and Lynch 1994), we define (probabilistic) bisimulation and simulation.

Definition 2.5. (cf. (Larsen and Skou 1991; Segala and Lynch 1994)) Let $(S, \rightarrow)$ be a transition system. A *bisimulation* on S is an equivalence relation R on S such that for all $(s, s') \in R$:

$$\text{If } s \xrightarrow{\alpha} \mu \text{ then there is a transition } s' \xrightarrow{\alpha} \mu' \text{ with } \mu[A] = \mu'[A] \text{ for all } A \in S/R$$

where S/R denotes the quotient space. Two states s and s' are called *bisimilar* (denoted by $s \sim s'$) iff there exists a bisimulation which contains (s, s').

The result of Milner (Milner 1989) that in every image-finite non-probabilistic transition system bisimulation can be approximated by "finitary bisimulation" carries over to (probabilistic) transition systems. Let $(S, \rightarrow)$ be a transition system and define inductively equivalence relations $\sim_n$ on S. We set $\sim_0 = S \times S$ and $s \sim_{n+1} s'$ iff:

— If $s \xrightarrow{\alpha} \mu$ then there is a transition $s' \xrightarrow{\alpha} \mu'$ with $\mu[A] = \mu'[A]$ for all $A \in S/\sim_n$.
— If $s' \xrightarrow{\alpha} \mu'$ then there is a transition $s \xrightarrow{\alpha} \mu$ with $\mu[A] = \mu'[A]$ for all $A \in S/\sim_n$.

Lemma 2.6. Let $(S, \rightarrow)$ be an image-finite transition system and $s, s' \in S$. Then, $s \sim s'$ iff $s \sim_n s'$ for all $n \geq 0$.

Proof. Let $\sim' = \bigcap_{n \geq 0} \sim_n$. We have to show that $\sim = \sim'$. It is easy to see that $\sim'$ is an equivalence relation. By induction on n it can be shown that $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \ldots \supseteq \sim$. Hence, $\sim' \supseteq \sim$. In order to show that $\sim' \subseteq \sim$ we show that $\sim'$ is a bisimulation.

For each $n \geq 0$ and each $A \in S/\sim'$, there exists a unique element $A_n \in S/\sim_n$ with $A \subseteq A_n$. Then, $A_0 = S \supseteq A_1 \supseteq A_2 \supseteq \ldots$ and $A = \bigcap A_n$.

Claim 1: If $s \xrightarrow{\alpha} \mu$ and $A \in S/\sim'$ then $\mu[A] = \inf_{n \geq 0} \mu[A_n]$.

Proof of Claim 1: Since $A = \bigcap A_n$ and $A_n \supseteq A_{n+1}$ we have $1 = \mu[A_0] \geq \mu[A_1] \geq \ldots \geq \mu[A]$. Let $r \stackrel{\text{def}}{=} \inf_{n \geq 0} \mu[A_n]$. Then, $r \geq \mu[A]$. We suppose $r > \mu[A]$. Then, $\Delta \stackrel{\text{def}}{=} r - \mu[A] > 0$. There exists a finite subset X of $S \setminus A$ such that $\mu[Y] < \Delta$ where $Y = S \setminus (A \cup X)$. (This is because $\sum_{t \notin A} \mu(t)$ is convergent.) For all $n \geq 0$, $A_n = A \cup (Y \cap A_n) \cup (X \cap A_n)$. The sets A, $Y \cap A_n$ and $X \cap A_n$ are pairwise disjoint. Hence,

$$\mu[A_n] = \mu[A] + \mu[Y \cap A_n] + \mu[X \cap A_n] < \mu[A] + \Delta + \mu[X \cap A_n] = r + \mu[X \cap A_n].$$

Since $r \leq \mu[A_n]$ we get $X \cap A_n \neq \emptyset$. Since X is finite and $A_0 \supseteq A_1 \supseteq \ldots$ we get $X \cap A \neq \emptyset$. Contradiction! (Note that, by definition, X is a subset of $S \setminus A$.)

Claim 2: $\sim'$ is a bisimulation.

Proof of Claim 2: Let $s \sim' s'$ and $s \xrightarrow{\alpha} \mu$. By Claim 1 it suffices to show that there is a transition $s' \xrightarrow{\alpha} \mu'$ such that $\mu[C] = \mu'[C]$ for all $n \geq 1$ and $C \in S/\sim_n$.

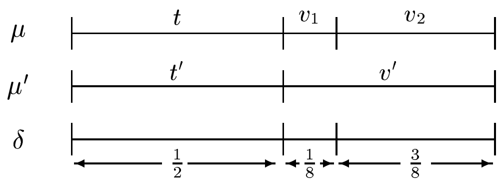
Since $s \sim_n s'$ there exist transitions $s' \xrightarrow{\alpha} \mu'_n$ such that $\mu[C] = \mu'_n[C]$ for all $C \in S/\sim_n$. Since $(S, \rightarrow)$ is image-finite the set $\{\mu'_n : n \geq 0\}$ is finite. Hence, there exists a transition $s' \xrightarrow{\alpha} \mu'$ with $\mu' = \mu'_n$ for infinitely many n. Let $n \geq 1$ and $C \in S/\sim_n$. There exists $k \geq n$ with $\mu' = \mu'_k$. C is of the form $C = \bigcup_{i \in I} B_i$ where $(B_i)_{i \in I}$ is a family of pairwise disjoint equivalence classes w.r.t. $\sim_k$ (as $\sim_k \subseteq \sim_n$). Thus, $\mu[B_i] = \mu'_k[B_i] = \mu'[B_i]$ for all $B \in S/\sim_k$. Then, $\mu[C] = \sum_{i \in I} \mu[B_i] = \sum_{i \in I} \mu'[B_i] = \mu'[C]$. $\square$

The following characterization of bisimulation goes back to (Jonsson and Larsen 1991). To be precise, the model considered there is based on labelled Markov chains rather than probabilistic transition systems in our sense. However, it can be easily seen that the results of (Jonsson and Larsen 1991) can be adapted to our setting.

Lemma 2.7. (cf. Theorem 4.6 in (Jonsson and Larsen 1991)) Let $(S, \rightarrow)$ be a probabilistic transition system and $s, s' \in S$. Then, $s \sim s'$ if and only if there exists an equivalence relation R on S which contains (s, s') and satisfies:

If $(t, t') \in R$ and $t \xrightarrow{\alpha} \mu$ then there is a transition $t' \xrightarrow{\alpha} \mu'$ with $\mu \preceq_R \mu'$.

Example 2.8. The states s and s' in the system shown in Figure 1 are bisimilar. A weight function for (μ, μ') w.r.t. $\sim$ can be obtained by combining parts of bisimilar states. Clearly, $t \sim t'$ and $v_1, v_2 \sim v'$.



Thus, $\delta(t, t') = 1/2$, $\delta(v_1, v') = 1/8$, $\delta(v_2, v') = 3/8$ (and $\delta(x, y) = 0$ in all other cases) yields a weight function for (μ, μ') w.r.t. $\sim$. ■

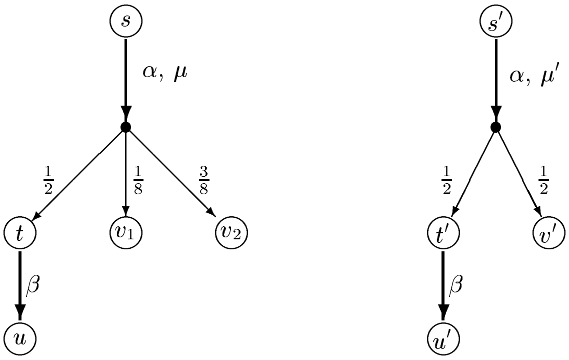


Fig. 1. $s \sim s'$

As in the non-probabilistic case, simulation can be viewed as "uni-directional bisimulation" in the sense that a process $\mathcal{P}'$ "simulates" another process $\mathcal{P}$ if each step of $\mathcal{P}$ can be "simulated" by a step of $\mathcal{P}'$. In that case, $\mathcal{P}$ can be viewed as an "implementation" of $\mathcal{P}'$ as each step of $\mathcal{P}$ is "allowed" by the "specification" $\mathcal{P}'$. The notion of a simulation can be obtained from the notion of a bisimulation by dropping the symmetry. Formally, in the characterization of a bisimulation in Lemma 2.7 we give up the requirement R to be an equivalence relation.

Definition 2.9. (cf. (Segala and Lynch 1994)) Let $(S, \rightarrow)$ be a probabilistic transition system. A *simulation* for $(S, \rightarrow)$ is a subset R of $S \times S$ such that for all $(s, s') \in R$:

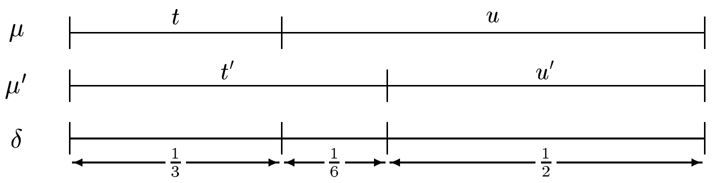
$$\text{If } s \xrightarrow{\alpha} \mu \text{ then there is a transition } s' \xrightarrow{\alpha} \mu' \text{ with } \mu \preceq_R \mu'.$$

We say s *implements* s' and s' *simulates* s (denoted by $s \sqsubseteq_{\text{sim}} s'$) iff there exists a simulation which contains (s, s'). s and s' are called *similar* (denoted by $s \sim_{\text{sim}} s'$) iff $s \sqsubseteq_{\text{sim}} s'$ and $s' \sqsubseteq_{\text{sim}} s$.

In the non-probabilistic case the above notion of a simulation agrees with Milner's notion of a simulation (Milner 1989). This is because the only weight function for (μ, μ'), where $\mu = \mu_s^1$, $\mu' = \mu_{s'}^1$ are simple distributions, is $\delta(u, u') = 0$ if $(u, u') \neq (s, s')$ and $\delta(s, s') = 1$. Hence, if $(S, \rightarrow)$ is a non-probabilistic transition system and $R \subseteq S \times S$ then

R is a simulation in the sense of Definition 2.9 if and only of R is a simulation in the sense of Milner (Milner 1989).

Example 2.10. Consider the transition system of Figure 2. Clearly, $u \sqsubseteq_{\mathrm{sim}} u'$ and $u, t \sqsubseteq_{\mathrm{sim}} t'$. A weight function for (μ, μ') can be obtained by combining certain parts of t (of u) with certain parts of t' (of u' and t'). The weight function δ for (μ, μ') w.r.t. $\sqsubseteq_{\mathrm{sim}}$ is given by: $\delta(t, t') = 1/3$, $\delta(u, t') = 1/6$, $\delta(u, u') = 1/2$.



We obtain $s \sqsubseteq_{\mathrm{sim}} s'$. ■

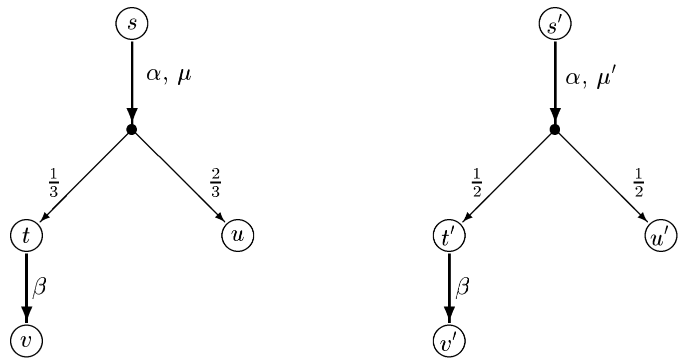


Fig. 2. $s \sqsubseteq_{\mathrm{sim}} s'$

We define "finitary simulation" $\sqsubseteq_n$ as follows. Let $(S, \rightarrow)$ be a probabilistic transition system. By induction on n we define relations $\sqsubseteq_n \subseteq S \times S$:

— $s \sqsubseteq_0 s'$ for all states s, $s' \in S$

— $s \sqsubseteq_{n+1} s'$ iff whenever $s \xrightarrow{\alpha} \mu$ then there exists a transition $s' \xrightarrow{\alpha} \mu'$ and a weight function δ for (μ, μ') w.r.t. $\sqsubseteq_n$ (i.e. $\mu \preceq_{\sqsubseteq_n} \mu'$).

Similarly to Lemma 2.6 we obtain the following.

Lemma 2.11. Let $(S, \rightarrow)$ be an image-finite probabilistic transition system and s, $s' \in S$. Then, $s \sqsubseteq_{\mathrm{sim}} s'$ if and only if $s \sqsubseteq_n s'$ for all $n \geq 0$.

Proof. Let $\sqsubseteq' = \bigcap \sqsubseteq_n$. By induction on n it can be shown that $\sqsubseteq_0 \supseteq \sqsubseteq_1 \supseteq \ldots \supseteq \sqsubseteq_{\mathrm{sim}}$. Hence, $\sqsubseteq' \supseteq \sqsubseteq_{\mathrm{sim}}$. In order to prove that $\sqsubseteq' \subseteq \sqsubseteq_{\mathrm{sim}}$ we show that $\sqsubseteq'$ is a simulation.

Let $s \sqsubseteq' s'$ and $s \xrightarrow{\alpha} \mu$. There exist transitions $s' \xrightarrow{\alpha} \mu'_n$ and weight functions δ_n for (μ, μ'_n) w.r.t. $\sqsubseteq_{n-1}$. Because of the image-finiteness we obtain that there exists a transition $s' \xrightarrow{\alpha} \mu'$ such that $\mu' = \mu'_n$ for infinitely many n. Let A_n be the set of pairs $(u, u') \in S \times S$ with $\delta_n(u, u') \neq 0$. Let B_k, $B'_k \subseteq S$, be finite sets with

$$\sum_{s \in B_k} \mu(s) \;>\; 1 - \frac{1}{2^k}, \quad \sum_{s' \in B'_k} \mu'(s') \;>\; 1 - \frac{1}{2^k}.$$

W.l.o.g. $B_1 \subseteq B_2 \subseteq \ldots$, $B'_1 \subseteq B'_2 \subseteq \ldots$. We define by induction on n infinite sets $I_0 \supseteq I_1 \supseteq \ldots$ of natural numbers such that the sequence $(\delta_n(u, u'))_{n \in I_i}$ is convergent for all $(u, u') \in B_i \times B'_i$, $i \geq 1$.

We put $I_0 \stackrel{\text{def}}{=} \{n : \mu'_n = \mu\}$. Let $i \geq 1$. We suppose that I_{i-1} is already defined. Let $(u_1, u'_1), \ldots, (u_k, u'_k)$ be the sequence of pairwise distinct pairs $(u, u') \in B_i \times B'_i$ which do not belong to $B_{i-1} \times B'_{i-1}$. For all infinite sets J of natural numbers and each $l \in \{1, \ldots, k\}$, there exists an infinite subset $J(l)$ of J such that $(\delta_n(u_l, u'_l))_{n \in J(l)}$ is convergent.† We define $J_1 \stackrel{\text{def}}{=} I_{i-1}(1)$, $J_l \stackrel{\text{def}}{=} J_{l-1}(l)$, $l = 2, \ldots, k$ and $I_i \stackrel{\text{def}}{=} J_k$. Then, I_i is an infinite subset of I_{i-1} and $(\delta_n(u, u'))_{n \in I_i}$ is convergent for all $(u, u') \in B_i \times B'_i$. We put

$$\delta : S \times S \to [0, 1], \quad \delta(u, u') \;\stackrel{\text{def}}{=}\; \lim_{\substack{n \to \infty \\ n \in I_i}} \delta_n(u, u')$$

if $(u, u') \in B_i \times B'_i$ and $\delta(u, u') = 0$ if $(u, u') \notin B_i \times B'_i$ for all $i \geq 1$. We show that δ is a weight function for (μ, μ') w.r.t. $\sqsubseteq'$.

1. If $\delta(u, u') > 0$ then (u, u') is contained in the countable set $\bigcup B_i \times B'_i$.
2. Let $u \in S$. We show that $\sum_{u'} \delta(u, u') = \mu(u)$. If $\mu(u) = 0$ then $\delta_n(u, u') = 0$ for all $u' \in S$ and $n \geq 0$. Hence, $\delta(u, u') = 0$ for all $u' \in S$. We suppose $\mu(u) > 0$. Let i be the smallest natural number i such that $1/2^i < \mu(u)$. Then, $u \in B_j$ for all $j \geq i$. Let A' be a finite subset of S. There exists some $j \geq i$ such that $\{u' \in A' : \mu'(u') > 0\} \subseteq B'_j$. Thus,

$$\sum_{u' \in A'} \delta(u, u') \;=\; \lim_{\substack{n \to \infty \\ n \in I_j}} \sum_{u' \in A'} \delta_n(u, u') \;\leq\; \mu(u)$$

Hence, $\sum_{u'} \delta(u, u') \;\leq\; \mu(u)$. In order to show that $\sum_{u'} \delta(u, u') \;\geq\; \mu(u)$ it is sufficient to show that for all $\varepsilon > 0$ there exists a finite subset A' of S with

$$\sum_{u' \in A'} \delta(u, u') \;\;\geq\;\; \mu(u) - \varepsilon.$$

Let $\varepsilon > 0$ and $j \geq i$ with $1/2^j \;<\; \varepsilon$. For all $n \in I_j$,

$$\sum_{u' \notin B'_j} \delta_n(u, u') \;\leq\; \sum_{u' \notin B'_j} \mu'(u') \;=\; 1 \;-\; \sum_{u' \in B'_j} \mu'(u') \;<\; \frac{1}{2^j} \;<\; \varepsilon.$$

† This is because $\delta_n(u_l, u'_l) \in [0, 1]$, and hence $(\delta_n(u_l, u'_l))_{n \in J}$ is bounded. Therefore, it contains a convergent subsequence.

Hence,

$$\sum_{u' \in B'_j} \delta_n(u,u') \;=\; \mu(u) \;-\; \sum_{u' \notin B'_j} \delta_n(u,u') \;>\; \mu(u) - \varepsilon$$

for all $n \in I_j$. Therefore,

$$\sum_{u' \in B'_j} \delta(u,u') \;=\; \lim_{\substack{n\to\infty \\ n \in I_j}} \sum_{u' \in B'_j} \delta_n(u,u') \;\geq\; \mu(u) - \varepsilon.$$

Similarly, it can be shown that $\sum_u \; \delta(u,u') \;=\; \mu'(u')$.

3. If $\delta(u,u') > 0$ then $(u,u') \in B_i \times B'_i$ for some $i \geq 1$ and $\delta_n(u,u') > 0$ for infinitely many n (more precisely, for almost all $n \in I_i$). Hence, $u \;\sqsubseteq_n\; u'$ for infinitely many n, and therefore $u \;\sqsubseteq'\; u'$.

□

In what follows, we deal with *pointed probabilistic transition systems*, i.e. probabilistic transition systems with a distinguished state, called the *initial state*. Formally, a pointed probabilistic transition system is a tuple $\mathcal{P} = (S, \rightarrow, s_0)$ where $(S, \rightarrow)$ is a probabilistic transition system and $s_0 \in S$. Two pointed probabilistic transition systems are called (bi-)similar iff their initial states are (bi-)similar as states of the probabilistic transition system $(S \uplus S', \rightarrow \,\cup \rightarrow')$. Here, $\uplus$ means disjoint union. Each distribution μ on S is identified with the distribution $\overline{\mu} : S \uplus S' \rightarrow [0,1]$, $\overline{\mu}(t) = \mu(t)$ if $t \in S$ and $\overline{\mu}(t') = 0$ if $t' \in S'$. Similarly, each distribution μ' on S' is viewed as a distribution on $S \uplus S'$.

Given a probabilistic transition system $(S, \rightarrow)$ and two states s, $s' \in S$, it is easy to see that s and s' are (bi-)similar as states of $(S, \rightarrow)$ if and only if the induced pointed transition sytems $\mathcal{P} = (S, \rightarrow, s)$, $\mathcal{P}' = (S, \rightarrow, s')$ are (bi-)similar. We extend $\sim_n$ and $\sqsubseteq_n$ to pointed transition systems by putting $\mathcal{P} \sim_n \mathcal{P}'$ iff $s \sim_n s'$ ($\mathcal{P} \sqsubseteq_n \mathcal{P}'$ iff $s \sqsubseteq_n s'$), where s and s' are considered as states of the probabilistic transition system $(S \uplus S', \rightarrow \cup \rightarrow')$. Lemma 2.6 and 2.11 allow us to prove the following characterization.

Lemma 2.12. Let $\mathcal{P}$ and $\mathcal{P}'$ be image-finite pointed probabilistic transition systems.
(a) $\mathcal{P} \sim \mathcal{P}'$ iff $\mathcal{P} \sim_n \mathcal{P}'$ for all $n \geq 0$.
(b) $\mathcal{P} \sqsubseteq_{\rm sim} \mathcal{P}'$ iff $\mathcal{P} \sqsubseteq_n \mathcal{P}'$ for all $n \geq 0$.

The alternative characterization of bisimilarity based on weight functions (Lemma 2.7) adapted for the pointed case is as follows.

Lemma 2.13. Let $\mathcal{P} = (S, \rightarrow, s)$, $\mathcal{P}' = (S', \rightarrow', s')$ be pointed probabilistic transition systems. Then, $\mathcal{P} \sim \mathcal{P}'$ iff there exists a relation $R \subseteq S \times S'$ which contains (s, s') and satisfies:

— If $(t,t') \in R$ and $t \stackrel{\alpha}{\rightarrow} \mu$ then there is a transition $t' \stackrel{\alpha}{\rightarrow}' \mu'$ with $\mu \preceq_R \mu'$.
— If $(t,t') \in R$ and $t' \stackrel{\alpha}{\rightarrow}' \mu'$ then there is a transition $t \stackrel{\alpha}{\rightarrow} \mu$ with $\mu \preceq_R \mu'$.

The following lemma shows that $\sqsubseteq_{\rm sim}$ is a preorder and its kernel $\sim_{\rm sim}$ is coarser than bisimulation equivalence.

Lemma 2.14. Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}'_1$, $\mathcal{P}'_2$ be pointed probabilistic transition systems.

(a) $\mathcal{P}_1 \sim \mathcal{P}_2 \implies \mathcal{P}_1 \sim_{\text{sim}} \mathcal{P}_2$
(b) $\mathcal{P}_1 \sqsubseteq_{\text{sim}} \mathcal{P}_2$, $\mathcal{P}_2 \sqsubseteq_{\text{sim}} \mathcal{P}_3 \implies \mathcal{P}_1 \sqsubseteq_{\text{sim}} \mathcal{P}_3$
(c) $\mathcal{P}_1 \sqsubseteq_{\text{sim}} \mathcal{P}_2$, $\mathcal{P}_1 \sim \mathcal{P}_1'$, $\mathcal{P}_2 \sim \mathcal{P}_2' \implies \mathcal{P}_1' \sqsubseteq_{\text{sim}} \mathcal{P}_2'$

Proof. (a) follows immediately by the definition of a simulation and Lemma 2.7. (c) follows by (a) and (b). The transitivity of $\sqsubseteq_{\text{sim}}$ (item (b)) can be derived from Remark 2.2. □

By Lemma 2.14(a) we have $\sim\ \subseteq\ \sim_{\text{sim}}$. As in the non-probabilistic case, simulation equivalence $\sim_{\text{sim}}$ does not coincide with bisimulation equivalence $\sim$. For instance, for the non-probabilistic pointed transition systems $\mathcal{P}$, $\mathcal{P}'$ (with initial states s and s' respectively) of Figure 3 we have $\mathcal{P} \sim_{\text{sim}} \mathcal{P}'$ but $\mathcal{P} \not\sim \mathcal{P}'$. In the case of reactive transition systems, simulation and bisimulation equivalence coincide, as shown in Appendix A.1, Theorem A.6. This result can be viewed as the probabilistic counterpart to the well-known result that simulation equivalence and bisimulation equivalence are the same for "deterministic" (non-probabilistic) transition systems.

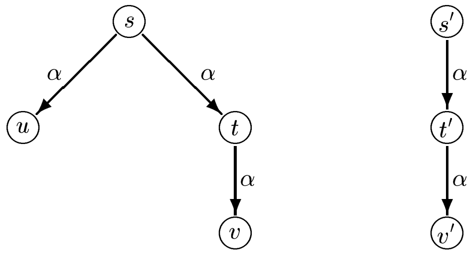


Fig. 3. $s \sim_{\text{sim}} s'$ but $s \not\sim s'$

Theorem 2.15. Let $\mathcal{P}$, $\mathcal{P}'$ be reactive transition systems. Then, $\mathcal{P} \sim_{\text{sim}} \mathcal{P}'$ iff $\mathcal{P} \sim \mathcal{P}'$.

3. Domain equations for probabilistic processes

We now turn our attention to the construction of semantic domains which can serve as fully abstract, denotational, models for languages with probabilistic transitions and recursion. Full abstraction w.r.t. bisimulation means that the semantic domain identifies exactly those pointed transition system which are bisimilar. Full abstraction w.r.t. simulation requires that the semantic domain be equipped with a partial order which reflects the simulation preorder $\sqsubseteq_{\text{sim}}$. As in the non-probabilistic case, where fully abstract models for bisimulation and simulation are given in terms of labelled trees, e.g. (Abramsky 1991; de Bakker and Meyer 1988; de Bakker and Zucker 1982; Golson and Rounds 1983), we use the concept of *probabilistic trees*, i.e. pointed transition systems which can be depicted as rooted trees (i.e. each state is reachable by a unique path from the initial state). For simplicity, we restrict ourselves to finitely branching probabilistic transition systems. We define the semantic domains by recursive equations soluble in suitable categories of complete metric spaces and domains by the methods of (Rutten and Turi 1993), resp. (Abramsky and Jung 1994). The central idea is to represent a probabilistic tree by a set of pairs (α, μ) where α is an action and μ a distribution on probabilistic trees. More formally, our aim is to solve domain equations of the form

$$X \;\cong\; \wp_*(Act \times \mathcal{D}_1(X))$$

where $\wp_*(\cdot)$ denotes a suitable powerdomain construction. Unfortunately, in both cases the equation cannot be solved by the methods of (America and Rutten 1989), resp. (Abramsky and Jung 1994), since the operator $\mathcal{D}_1(\cdot)$ does not preserve completeness (cf. Remark 4.6 and 4.9). Nevertheless, the equation $X \;\cong\; \wp_{\text{fin}}(Act \times \mathcal{D}_1(X))$ has a final solution in *SET*, the category of sets and functions (see Section 3.1). Here, $\wp_{\text{fin}}(\cdot)$ denotes the collection of finite subsets of $(\cdot)$. Therefore, we get a "final semantics" for probabilistic transition systems in the sense of (Rutten and Turi 1993). We switch from $\mathcal{D}_1(\cdot)$ to the probabilistic powerdomain $\mathcal{E}_1(\cdot)$ of probabilistic evaluations in the sense of (Jones and Plotkin 1989), that is, we solve domain equations of the form

$$X \;\cong\; \wp_*(Act \times \mathcal{E}_1(X))$$

by the methods of (Rutten and Turi 1993; Abramsky and Jung 1994) obtaining semantic domains which subsume probabilistic trees (see Section 4.2 and 4.3). In the metric aproach we work with the powerdomain operator $\wp_{\text{comp}}(\cdot)$ which assigns to each metric space M the set of compact subsets of M. When working with domains we use the Hoare powerdomain $\wp_{\text{Hoare}}(\cdot)$.

3.1. *Probabilistic trees*

We show that the bisimulation equivalence classes of finitely branching pointed probabilistic transition systems can be considered as *probabilistic trees*, and that they form the final solution of the domain equation $X \;\cong\; \wp_{\text{fin}}(Act \times \mathcal{D}_1(X))$.

Definition 3.1. Let *ProbTree* be the set of bisimulation equivalence classes of finitely branching pointed probabilistic transition systems. We refer to the elements of *ProbTree* as *probabilistic trees*. Whenever $\mathcal{P}$ is a finitely branching pointed probabilistic transition system then *probtree*$(\mathcal{P})$ denotes the bisimulation equivalence class of $\mathcal{P}$.

The notion of a "probabilistic tree" for a bisimulation equivalence class is justified since each bisimulation class contains a finitely branching, pointed, probabilistic transition system where the underlying directed graph is a tree in the graph-theoretic sense. Moreover, in each bisimulation class there is a "unique" finitely branching pointed probabilistic transition system whose underlying directed graph is a tree, and whenever s is a state of it and $s \xrightarrow{\alpha} \mu$, $s \xrightarrow{\alpha} \mu'$ with $\mu[A] = \mu'[A]$ for all bisimulation equivalence classes A then $\mu = \mu'$. Here, "uniqueness" is up to isomorphism, i.e. up to the names of the states.

Remark 3.2. In order to see that *ProbTree* is really a set consider a fixed set S_0 of cardinality ω and define *ProbTree* to be the set of bisimulation equivalence classes of finitely branching pointed transition systems whose states belong to S_0. Then, each finitely branching pointed transition system is bisimilar to some finitely branching pointed probabilistic transition system whose states belong to S_0. This is because the set of states s which can be reached from the initial state is always countable. ■

In what follows, by a pointed transition system we mean a finitely branching pointed probabilistic transition system. We associate with *ProbTree* a probabilistic transition system $(\mathit{ProbTree}, \rightarrow)$ as follows. If $\mathcal{T}$ is the bisimulation equivalence class of a pointed transition system $\mathcal{P} = (S, \rightarrow, s)$ then we put $\mathcal{T} \stackrel{\alpha}{\rightarrow} \nu$ iff there exists a transition $s \stackrel{\alpha}{\rightarrow} \mu$ with $\nu(\mathcal{T}_A) = \mu[A]$ for all bisimulation equivalence classes $A \in S/\sim$. Here, $\mathcal{T}_A$ is the unique probabilistic tree which represents A, i.e. $\mathcal{T}_A = \mathit{probtree}(S, \rightarrow, t)$ for all (some) $t \in A$. Thus, if $\mathcal{P} = (S, \rightarrow, s)$ then $\mathit{probtree}(\mathcal{P}) \stackrel{\alpha}{\rightarrow} \nu$ iff there exists a transition $s \stackrel{\alpha}{\rightarrow} \mu$ such that $\nu = \mathcal{D}_1(\mathit{probtree} \circ \varpi)(\mu)$ where ϖ is the function which assigns to each state $t \in S$ the pointed transition system $\varpi(t) \stackrel{\mathrm{def}}{=} (S, \rightarrow, t)$. Each element $\mathcal{T}$ of *ProbTree* is identified with the pointed transition system $(\mathit{ProbTree}, \rightarrow, \mathcal{T})$. The following theorem and its corollary show that for each pointed transition system $\mathcal{P}$, $\mathit{probtree}(\mathcal{P})$ (viewed as a pointed transition system) is the unique probabilistic tree which is bisimilar to $\mathcal{P}$.

Theorem 3.3. Let $\mathcal{P}$, $\mathcal{P}'$ be pointed transition system.
(a) $\mathcal{P} \sim \mathit{probtree}(\mathcal{P})$.
(b) $\mathcal{P} \sim \mathcal{P}'$ if and only if $\mathit{probtree}(\mathcal{P}) = \mathit{probtree}(\mathcal{P}')$.
(c) $\mathcal{P} \sqsubseteq_{\mathrm{sim}} \mathcal{P}'$ if and only if $\mathit{probtree}(\mathcal{P}) \sqsubseteq_{\mathrm{sim}} \mathit{probtree}(\mathcal{P}')$.

Proof. (b) is clear since $\mathit{probtree}(\cdot)$ is defined to be the bisimulation equivalence class of $(\cdot)$. (c) follows by (a) and Lemma 2.14(c). We show (a). Let $\mathcal{P} = (S, \rightarrow, s)$. With each state t of S we associate the pointed transition system $\mathcal{P}_t \stackrel{\mathrm{def}}{=} (S, \rightarrow, t)$ and put $R \stackrel{\mathrm{def}}{=} \{(t, \mathit{probtree}(\mathcal{P}_t)) : t \in S\}$. We show that R fulfills the conditions of Lemma 2.13. Let $t \in S$. Let $f : S \rightarrow \mathit{ProbTree}$ be given by $f(t) \stackrel{\mathrm{def}}{=} \mathit{probtree}(\mathcal{P}_t)$. It is easy to see that:
— If $t \stackrel{\alpha}{\rightarrow} \mu$ then $\mathit{probtree}(\mathcal{P}_t) \stackrel{\alpha}{\rightarrow} \mathcal{D}_1(f)(\mu)$.
— If $\mathit{probtree}(\mathcal{P}_t) \stackrel{\alpha}{\rightarrow} \nu$ then $\nu = \mathcal{D}_1(f)(\mu)$ for some transition $s \stackrel{\alpha}{\rightarrow} \mu$.
By Remark 2.4, $\mu \preceq_R \mathcal{D}_1(f)(\mu)$. By Lemma 2.13, $\mathcal{P}_t \sim f(t) = \mathit{probtree}(\mathcal{P}_t)$ for all $t \in S$. □

Corollary 3.4. Let $\mathcal{T}$, $\mathcal{T}' \in \mathit{ProbTree}$. Then, $\mathcal{T} \sim \mathcal{T}'$ iff $\mathcal{T} = \mathcal{T}'$.

Proof. Let $\mathcal{P}$, $\mathcal{P}'$ be pointed transition systems such that $\mathcal{T} = \mathit{probtree}(\mathcal{P})$, $\mathcal{T}' = \mathit{probtree}(\mathcal{P}')$. Then, by Lemma 3.3(a), $\mathcal{T} \sim \mathcal{P}$ and $\mathcal{T}' \sim \mathcal{P}'$. Hence, $\mathcal{T} \sim \mathcal{T}'$ implies $\mathcal{P} \sim \mathcal{P}'$, and therefore $\mathcal{T} = \mathit{probtree}(\mathcal{P}) = \mathit{probtree}(\mathcal{P}') = \mathcal{T}'$. □

3.2. *A domain equation for probabilistic trees*

We show that probabilistic trees form a final solution of the equation $X \cong \wp_{\mathrm{fin}}(\mathit{Act} \times \mathcal{D}_1(X))$ in *SET*, the category of sets and functions. In Section 2.1 the definition of the functor $\mathcal{D}_1 : \mathit{SET} \rightarrow \mathit{SET}$ is given. If X is a set then $\wp_{\mathrm{fin}}(X)$ denotes the set of finite subsets of X and $\mathcal{A}ct(X) \stackrel{\mathrm{def}}{=} \mathit{Act} \times X$. If $f : X \rightarrow Y$ is a function then we define

$\wp_{\text{fin}}(f) : \wp_{\text{fin}}(X) \to \wp_{\text{fin}}(Y)$ and $\mathcal{A}ct(f) : \mathcal{A}ct(X) \to \mathcal{A}ct(Y)$ by $\wp_{\text{fin}}(f)(U) \stackrel{\text{def}}{=} f(U)$, $\mathcal{A}ct(f)(\alpha, x) \stackrel{\text{def}}{=} (\alpha, f(x))$. Hence, $\wp_{\text{fin}}$ and $\mathcal{A}ct$ can be considered as endofunctors of SET.

Theorem 3.5. *ProbTree* is the final coalgebra (and hence the final fixed point) of the functor $\wp_{\text{fin}} \circ \mathcal{A}ct \circ \mathcal{D}_1 : SET \to SET$.

Proof. Let $\mathcal{K} \stackrel{\text{def}}{=} \wp_{\text{fin}} \circ \mathcal{A}ct \circ \mathcal{D}_1$. We define $e : \mathit{ProbTree} \to \mathcal{K}(\mathit{ProbTree})$ by $e(\mathcal{T}) \stackrel{\text{def}}{=} \{(\alpha, \mu) : \mathcal{T} \stackrel{\alpha}{\to} \mu\}$. Then, $(\mathit{ProbTree}, e)$ is a coalgebra of $\mathcal{K}$. Let (Y, k) be a coalgebra of $\mathcal{K}$, i.e. $k : Y \to \mathcal{K}(Y)$ is a function. We associate with Y a finitely branching probabilistic transition system $(Y, \to)$ as follows: $y \stackrel{\alpha}{\to} \nu$ iff $(\alpha, \nu) \in k(y)$. For $y \in Y$, $\mathcal{P}_y$ denotes the pointed transition system $(Y, \to, y)$. It is easy to see that the function $F : Y \to \mathit{ProbTree}$, $F(y) \stackrel{\text{def}}{=} \mathit{probtree}(\mathcal{P}_y)$ satisfies $\mathcal{K}(F) \circ k = e \circ F$. We show the uniqueness of F as a function $f : Y \to \mathit{ProbTree}$ satisfying $\mathcal{K}(f) \circ k = e \circ f$. Whenever $f : Y \to \mathit{ProbTree}$ is a function with $\mathcal{K}(f) \circ k = e \circ f$ then we show that $R \stackrel{\text{def}}{=} \{(y, f(y)) : y \in Y\}$ satisfies the conditions of Lemma 2.13. It is easy to see that

— If $y \stackrel{\alpha}{\to} \nu$ then $f(y) \stackrel{\alpha}{\to} \mathcal{D}_1(f)(\nu)$.
— If $f(y) \stackrel{\alpha}{\to} \mu$ then $y \stackrel{\alpha}{\to} \nu$ for some $\nu \in \mathcal{D}_1(Y)$ where $\mu = \mathcal{D}_1(f)(\nu)$.

By Remark 2.4, $\nu \preceq_R \mathcal{D}_1(f)(\nu)$. Thus, $\mathcal{P}_y \sim f(y)$ for all $y \in Y$. By Theorem 3.3 and Corollary 3.4, $f(y) = \mathit{probtree}(\mathcal{P}_y)$ for all $y \in Y$. Hence, $F = f$. □

3.3. *Final semantics*

Since *ProbTree* is the final coalgebra we get a "final semantics" in the sense of (Rutten and Turi 1993). Let $(S, \to)$ be a finitely branching probabilistic transition system. Then, (S, k) is a coalgebra of $\mathcal{K} \stackrel{\text{def}}{=} \wp_{\text{fin}} \circ \mathcal{A}ct \circ \mathcal{D}_1$ where $k : S \to \mathcal{K}(S)$ is given by $k(s) \stackrel{\text{def}}{=} \{(\alpha, \nu) : s \stackrel{\alpha}{\to} \nu\}$. Let $F : S \to \mathit{ProbTree}$ be the unique function with $\mathcal{K}(F) \circ k = e \circ F$. As we saw in the proof of Theorem 3.5, $F(s) = \mathit{probtree}(s)$ where s is identified with the pointed transition system $(S, \to, s)$, i.e., having an operational semantics based on transition systems the induced "probabilistic tree semantics" (which assigns to each process the bisimulation equivalence class of its pointed transition system) coincides with the final semantics. By Theorem 3.3, the final semantics is fully abstract in the sense that it identifies two states if and only if they are bisimilar and that it preserves the simulation preorder $\sqsubseteq_{\text{sim}}$.

Example 3.6. We consider the pointed transition systems $\mathcal{P}$ and $\mathcal{P}'$ of Figure 2. (I.e. $\mathcal{P}$ and $\mathcal{P}'$ are the systems with initial state s and s' respectively.) The final semantics $\mathit{probtree}(\mathcal{P})$ and $\mathit{probtree}(\mathcal{P}')$ of $\mathcal{P}$ and $\mathcal{P}'$ are $\mathit{probtree}(\mathcal{P}) = \{(\alpha, \nu)\}$, $\mathit{probtree}(\mathcal{P}') = \{(\alpha, \nu')\}$ where $\nu(\mathcal{T}) = 1/3$, $\nu(\emptyset) = 2/3$, $\nu'(\mathcal{T}) = \nu'(\emptyset) = 1/2$ and $\mathcal{T} = \{(\beta, \mu^1_\emptyset)\}$, when viewed as elements of $\mathit{ProbTree} = \wp_{\text{fin}}(\mathcal{A}ct(\mathcal{D}_1(\mathit{ProbTree}))))$. ■

4. Denotational models for probabilistic trees

The denotational models of probabilistic processes (which subsume probabilistic trees) arise as solutions of the domain equation $X \cong \wp_*(Act \times \mathcal{E}_1(X))$ in appropriate categories of metric spaces and domains. Here, $\mathcal{E}_1(\cdot)$ denotes the probabilistic powerdomain in the sense of (Jones and Plotkin 1989; Jones 1990), and $\wp_*$ a suitably chosen powerdomain. We use the fixed point theorems of (Rutten and Turi 1993), which goes back to those of (America and Rutten 1989), and (Abramsky and Jung 1994), which goes back to those of (Smyth and Plotkin 1982)).

We suppose the reader to be familiar with basic notions concerning domain theory and metric spaces; for details see e.g. (Abramsky and Jung 1994; Gierz *et al.* 1980) and (Engelking 1989). We recall the basic definitions to prevent confusion. If D is a partially ordered set (we denote the order by $\sqsubseteq$) then for $y \in D$, $Y \subseteq D$:

$$y \uparrow \stackrel{\text{def}}{=} \{x \in D : y \sqsubseteq x\}, \quad y \downarrow \stackrel{\text{def}}{=} \{x \in D : x \sqsubseteq y\}$$

and $Y \uparrow \stackrel{\text{def}}{=} \bigcup_{y \in Y} y \uparrow$, $Y \downarrow \stackrel{\text{def}}{=} \bigcup_{y \in Y} y \downarrow$. Y is called *downward-closed* (resp. *upward-closed*) iff $Y = Y \downarrow$ (resp. $Y = Y \uparrow$). A *complete partial order* (*cpo* for short) is a partially ordered set with a bottom element (denoted by $\perp$) where each directed subset X has a least upper bound (denoted by $\bigsqcup X$). A subset A of a cpo is called *lub-closed* iff for every directed subset X of A we have $\bigsqcup X \in A$. We suppose D to be equipped with the Scott-topology whose closed sets are the downward-closed and lub-closed subsets of D.

Let x, y be elements of a cpo D. We say y *approximates* x iff for all directed subsets X of D, $x \sqsubseteq \bigsqcup X$ implies $y \sqsubseteq z$ for some $z \in X$. $x \Downarrow$ denotes the set of elements $y \in D$ such that y approximates x. A *basis* of a cpo D is a subset B of D such that for each $x \in D$ the set $B_x = B \cap x \Downarrow$ contains a directed subset with least upper bound x. A *continuous domain* is a cpo which has a basis. $CONT_\perp$ denotes the category of continuous domains and strict and continuous functions.

Recall that an ultrametric space is a pair (M, d) consiting of a set M and a function $d : M \times M \to \mathbb{R}$ such that for all x, y, $z \in M$: (1) $d(x, y) = 0$ iff $x = y$ (2) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ and (3) $d(x, y) = d(y, x)$.

A Cauchy sequence in an ultrametric space (M, d) is a sequence $(x_n)_{n \geq 0}$ in M where for any $\varepsilon > 0$ there is some $N \geq 0$ such that $d(x_n, x_m) \leq \varepsilon$ for all n, $m \geq N$. An ultramteric space (M, d) is called complete iff for any Cauchy sequence $(x_n)_{n \geq 0}$ the limit $\lim_{n \to \infty} x_n$ exists. Here, an element $x \in M$ is called the limit of $(x_n)_{n \geq 0}$ iff for all $\varepsilon > 0$ there is some $N \geq 0$ such that $d(x_n, x) \leq \varepsilon$ for all $n \geq N$.

Let $f : M_1 \to M_2$ be a function where (M_1, d_1), (M_2, d_2) are ultrametric spaces. f is called non-expansive iff $d_2(f(x), f(y)) \leq d_1(x, y)$ for all x, $y \in M_1$. f is called contractive iff there is a constant $C \in]0, 1[$ such that $d_2(f(x), f(y)) \leq C \cdot d_1(x, y)$ for all x, $y \in M_1$. In this case, C is called the contraction coefficient of f.

Banach's fixed point theorem states that any contracting self-mapping of a nonempty complete ultrametric (M, d) has a unique fixed point, denoted $\mathit{fix}(f)$, which is given by

$$\mathit{fix}(f) = \lim_{n \to \infty} f^n(x_0)$$

where $x_0 \in M$ is an arbitrary element of M and $f^0 = id_M$ and $f^n = f \circ f^{n-1}$, $n = 1, 2, \ldots$.

In the following, we often simply write M rather than (M, d), and suppose that the underlying distance on an ultrametric space – which we always denote by d – satisfies $d \leq 1$. The topology on an ultrametric space M is given by taking the open balls $B(x, \rho)$,

$x \in M$, $\rho > 0$, as its basic opens. $\mathcal{B}$ denotes the set of all open balls, $\mathcal{B}_\rho$ the set of open balls with radius $\geq \rho$, i.e. open balls of the form $B(x,r)$ where $r \geq \rho$. By a *ρ-set* we mean an open set $U \subseteq M$ such that $B(x,\rho) \subseteq U$ for all $x \in U$. *CUM* denotes the category of complete ultrametric spaces and non-expansive functions.

4.1. *Probabilistic powerdomains*

Following (Jones and Plotkin 1989) we define an *evaluation* on a topological space X as a function $E : \Omega(X) \to \mathbb{R}_{\geq 0}$ such that

1. Whenever $U \subseteq U'$ then $E(U) \leq E(U')$.
2. $E(U \cap U') + E(U \cup U') = E(U) + E(U')$.
3. $E(\emptyset) = 0$.

In the above, $\Omega(X)$ is the set of open sets in X. An evaluation E on X is said to be *continuous* iff for each directed family $(U_i)_{i \in I}$ of open sets:

$$E\left(\bigcup_{i \in I} U_i\right) = \sup_{i \in I} E(U_i)$$

$\mathcal{E}(X)$ denotes the set of all continuous evaluations on X. E is called a *probabilistic evaluation* iff $E(X) = 1$. The *probabilistic powerdomain* $\mathcal{E}_1(X)$ of X is the set of probabilistic continuous evaluations on X. In what follows, by an evaluation we mean a probabilistic continuous evaluation. We extend evaluations to closed subsets of X where we put $E(A) \stackrel{\text{def}}{=} 1 - E(X \setminus A)$ for each closed subset A of X. If X, X' are topological spaces and $f : X \to X'$ is a continuous function then we put:

$$\mathcal{E}_1(f) : \mathcal{E}_1(X) \to \mathcal{E}_1(X'), \quad \mathcal{E}_1(f)(E)(U) \stackrel{\text{def}}{=} E(f^{-1}(U))$$

$\mathcal{E}_1$ can be considered as a functor $TOP \to SET$ where TOP denotes the category of topological spaces and continuous functions. The following lemma shows that for each topological space X, the distributions on X induce evaluations on X.

Lemma 4.1. Let X be a topological space and μ a distribution on X. Then,

$$E_\mu : \Omega(X) \to [0,1], \quad E_\mu(U) \stackrel{\text{def}}{=} \mu[U]$$

is an evaluation on X.

Proof. Easy verification. $\square$

Whether the function $\mathcal{D}_1(X) \to \mathcal{E}_1(X)$, $\mu \mapsto E_\mu$, is injective (and hence can be considered as an embedding) depends on the underlying topology on X. Consider the topology $\{\emptyset, X\}$ on a set X which contains at least two points; it is easy to see that this function is not injective. In our applications – where X is equipped with an ultrametric or a complete partial order – $\mathcal{D}_1(X)$ can be considered as a subspace of $\mathcal{E}_1(X)$.

Remark 4.2. Let $eval_X : \mathcal{D}_1(X) \to \mathcal{E}_1(X)$ be the function $eval_X(\mu) \stackrel{\text{def}}{=} E_\mu$. It is easy to see that $eval_Y \circ \mathcal{D}_1(f) = \mathcal{E}_1(f) \circ eval_X$ for every continuous function $f : X \to Y$.

I.e., for each distribution $\mu \in \mathcal{D}_1(X)$, $E_{\mathcal{D}_1(f)(\mu)} = \mathcal{E}_1(f)(E_\mu)$. Hence, *eval* is a natural transformation $\mathcal{D}_1 \to \mathcal{E}_1$ where $\mathcal{D}_1$ is considered as a functor $TOP \to SET$. ■

One might expect that, for each continuous domain D, $\mathcal{D}_1(D)$ is a basis of $\mathcal{E}_1(D)$. In general, this is not the case, cf. Remark 4.16.

4.1.1. *Probabilistic powerdomains for cpo's* Along the lines of (Jones and Plotkin 1989) we define a locally continuous endofunctor on $CONT_\perp$ which assigns to each cpo D its probabilistic powerdomain $\mathcal{E}_1(D)$. Alternatively, we could work with $CPO_\perp$. Recall that we suppose a cpo D to be equipped with the Scott-topology:

$$\Omega(D) \;=\; \{\, U \subseteq D \;:\; U \text{ is upward-closed and } D \setminus U \text{ is lub-closed} \,\}.$$

In contrast to the approach of (Jones and Plotkin 1989) we deal with the powerdomain $\mathcal{E}_1(\cdot)$ of probabilistic evaluations instead of $\mathcal{E}(\cdot)$. If D is a cpo then $\mathcal{E}_1(D)$ is a cpo where we use the same partial order as in (Jones and Plotkin 1989):

$$E_1 \;\sqsubseteq\; E_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall\, U \in \Omega(U) :\; E_1(U) \;\leq\; E_2(U)$$

The bottom element $\perp_{\mathcal{E}_1(D)}$ of $\mathcal{E}_1(D)$ is $E_{\mu^1_{\perp_D}}$ (where $\perp_D$ is the bottom element of D), i.e. it is given by

$$U \;\mapsto\; \begin{cases} 1 & : \;\text{if } U = D, \\ 0 & : \;\text{otherwise.} \end{cases}$$

Thus, $\perp_{\mathcal{E}_1(D)}$ is different from the bottom element of $\mathcal{E}(D)$. Whenever $(E_i)_{i \in I}$ is a directed family of evaluations then the least upper bound $E = \bigsqcup E_i$ in $\mathcal{E}_1(D)$ is the least upper bound of $(E_i)_{i\in I}$ in $\mathcal{E}(D)$:

$$E(U) \;=\; \sup_{i \in I}\; E_i(U).$$

If D, D' are cpo's and $f : D \to D'$ is a strict continuous function then $\mathcal{E}_1(f)$ is strict and continuous. By the results of (Jones 1990), $\mathcal{E}(D)$ is continuous if D is continuous. It is easy to see that this result carries over to $\mathcal{E}_1(\cdot)$. Hence, $\mathcal{E}_1$ can be considered as a functor $CONT_\perp \to CONT_\perp$.

Lemma 4.3. The functor $\mathcal{E}_1 : CONT_\perp \to CONT_\perp$ is locally continuous.

Proof. We have to show that whenever $(f_i)_{i\in I}$ is a directed family of strict and continuous functions $f_i : D \to D'$ and $f = \bigsqcup f_i$ (where D and D' are cpo's) then $\mathcal{E}_1(f) \;=\; \bigsqcup \mathcal{E}_1(f_i)$. Let $E \in \mathcal{E}_1(D)$, $U \in \Omega(D')$. Then, $(f_i^{-1}(U))_{i\in I}$ is directed and $f^{-1}(U) \;=\; \bigcup f_i^{-1}(U)$. Since $\mathcal{E}_1(f)$ is continuous we get:

$$\mathcal{E}_1(f)(E)(U) \;=\; E(f^{-1}(U)) \;= \sup\; E(f_i^{-1}(U)) \;=\; \sup\; \mathcal{E}_1(f_i)(E)(U).$$

Hence, $\mathcal{E}_1(f)(E) \;=\; \bigsqcup \mathcal{E}_1(f_i)(E)$ for all $E \in \mathcal{E}_1(D)$. Thus, $\mathcal{E}_1(f) \;=\; \bigsqcup \mathcal{E}_1(f_i)$. □

Notation 4.4. Let D be a partially ordered set (with partial order $\sqsubseteq$) and $\mu, \mu' \in \mathcal{D}_1(D)$. Then, $\mu \preceq_{\text{sim}} \mu'$ iff there exists a weight function δ for (μ, μ') w.r.t. $\sqsubseteq$.[‡]

[‡] Using the notation of Definition 2.1, $\preceq_{\text{sim}} \;=\; \preceq_{\sqsubseteq}$.

In the Appendix (Section A.1, Theorem A.2 and Corollary A.4) we show the following result.

Theorem 4.5. If D is a cpo then $\preceq_{\text{sim}}$ is a partial order on $\mathcal{D}_1(D)$ and for all μ, $\mu' \in \mathcal{D}_1(D)$:

$$\mu \preceq_{\text{sim}} \mu' \iff E_\mu \sqsubseteq E_{\mu'}$$

In particular, the function $\mathcal{D}_1(D) \to \mathcal{E}_1(D)$, $\mu \mapsto E_\mu$, is order-preserving. Thus, each distribution μ can be identified with the evaluation E_μ.

Remark 4.6. In general, $\mathcal{D}_1(D)$ is not complete. Consider the cpo $D = \{0,1\}^\infty$ of all (finite or infinite) words built from 0 and 1 equipped with the prefix ordering. Let μ_k be the distribution with

$$\mu_k(x) = \begin{cases} 1/2^k & : \text{ if } x \text{ is a word of length } k \\ 0 & : \text{ otherwise.} \end{cases}$$

It is easy to see that $(\mu_k)_{k\geq 1}$ is a monotone sequence in $\mathcal{D}_1(D)$ which does not have an upper bound in $\mathcal{D}_1(D)$. In order to see that $\mu_k \sqsubseteq_{\text{sim}} \mu_{k+1}$ consider the distribution δ on $D \times D$ with $\delta(x, x0) = \delta(x, x1) = 1/2^{k+1}$ if x is a word of length k and $\delta(x, y) = 0$ in all other cases. ■

4.1.2. *Probabilistic powerdomains of ultrametric spaces* We show that the probabilistic powerdomain construction $\mathcal{E}_1(\cdot)$ can be considered as an endofunctor on *CUM* which is locally non-expansive in the sense of (Rutten and Turi 1993). Recall that for each ultrametric space M

$$\Omega(M) = \{ U \subseteq M : \forall x \in U \exists B \in \mathcal{B}. x \in B \subseteq U \}.$$

It is easy to see that for each distribution μ on M:

$$\mu(x) = \inf \{ E_\mu(U) : x \in U \in \Omega(M) \} = \inf \{ E_\mu(B) : x \in B \in \mathcal{B} \}.$$

By the above, whenever μ, $\mu' \in \mathcal{D}_1(M)$ with $E_\mu = E_{\mu'}$ then $\mu = \mu'$. Hence, the set $\mathcal{D}_1(M)$ of distributions on M can be considered as a subspace of $\mathcal{E}_1(M)$. We suppose $\mathcal{E}_1(M)$ to be endowed with the distance

$$d(E_1, E_2) \stackrel{\text{def}}{=} \inf \{ \rho > 0 : E_1(B) = E_2(B) \ \forall B \in \mathcal{B}_\rho \}.$$

From the results in the Appendix (Section A.2, Theorem A.12 and Theorem A.13) we have the following.

Theorem 4.7. If M is a complete ultrametric space then $\mathcal{E}_1(M)$ is a complete ultrametric space. In this case, $\mathcal{E}_1(M)$ is the completion of $\mathcal{D}_1(M)$ (considered as a subspace of $\mathcal{E}_1(M)$).

The ultrametric d on $\mathcal{E}_1(M)$ can be characterized as follows: $d(E_1, E_2) \leq r$ iff $E_1(B) = E_2(B)$ for all $B \in \bigcup_{\rho > r} \mathcal{B}_\rho$ iff $E_1(U) = E_2(U)$ for all ρ-sets U where $\rho > r$. Whenever $f : M \to M'$ is a non-expansive function between ultrametric spaces M and M' and B is an open ball in M' with radius ρ then $f^{-1}(B)$ is a ρ-set. From this, whenever E_1, E_2

are evaluations on $\mathcal{E}_1(M)$ then $d(\mathcal{E}_1(f)(E_1), \mathcal{E}_2(f)(E_2)) \leq d(E_1, E_2)$, i.e. the function $\mathcal{E}_1(f)$ is non-expansive. Hence, $\mathcal{E}_1$ can be considered as an endofunctor of *CUM*.

Lemma 4.8. The functor $\mathcal{E}_1 : CUM \to CUM$ is locally non-expansive in the sense of (Rutten and Turi 1993).

Proof. We have to show that for every pair (M, M') of complete ultrametric spaces the function $(M \to_{\text{nexp}} M') \to (\mathcal{E}_1(M) \to_{\text{nexp}} \mathcal{E}_1(M'))$, $f \mapsto \mathcal{E}_1(f)$, is non-expansive, where the distance on the function spaces $X \to_{\text{nexp}} X'$ (of non-expansive functions $X \to X'$) is given by

$$d(f_1, f_2) \stackrel{\text{def}}{=} \sup_{x \in X} d(f_1(x), f_2(x)).$$

Let $f_1, f_2 : M \to M'$ be non-expansive functions and let $d(f_1, f_2) = r$. Let E be an evaluation on M. We have to show that $d(\mathcal{E}_1(f_1)(E), \mathcal{E}_1(f_2)(E)) \leq r$. Let B be an open ball in M' with radius $\rho > r$. Since $d(f_1, f_2) = r$ we have $f_1^{-1}(B) = f_2^{-1}(B)$. Hence,

$$\mathcal{E}_1(f_1)(E)(B) \;=\; E(\; f_1^{-1}(B)\;) \;=\; E(\; f_2^{-1}(B)\;) \;=\; \mathcal{E}_1(f_2)(E)(B).$$

We conclude $d(\mathcal{E}_1(f_1), \mathcal{E}_1(f_2)) \leq r = d(f_1, f_2)$. □

Remark 4.9. Similarly to Remark 4.6 the set $\{0,1\}^\infty$ equipped with the natural distance

$$d(x, y) \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{2^n} : x[n] \neq y[n] \right\}$$

(where $z[n]$ denotes the n-th prefix of z) yields an example for a complete metric space M where $\mathcal{D}_1(M)$ is not complete (i.e. $\mathcal{D}_1(M)$ as a subspace of $\mathcal{E}_1(M)$ is not closed). Consider the sequence (μ_k) which is defined as in Remark 4.6. Then, $d(\mu_k, \mu_i) \leq 1/2^i$ for all $k \geq i$. Thus, (μ_k) is a Cauchy sequence in $\mathcal{D}_1(M)$ which does not have a limit in $\mathcal{D}_1(M)$. ■

4.2. *A domain-theoretic equation for probabilistic trees*

We now consider solving the equation $X \cong \wp_{\text{Hoare}}(Act \times \mathcal{E}_1(X))$ in a suitable category of domains. First, we have to demonstrate that it can indeed be solved with the help of the fixed point theorem of (Abramsky and Jung 1994; Smyth and Plotkin 1982) (here, $\wp_{\text{Hoare}}(\cdot)$ denotes the Hoare powerdomain of $(\cdot)$). If D is a cpo then $\wp_{\text{Hoare}}(D)$ is the cpo of nonempty and Scott-closed subsets of D ordered by inclusion. Recall that Scott-closedness means lub- and downward-closedness. For A to be a non-empty subset of D, A^* denotes the Scott-closure of A, i.e. the smallest Scott-closed subset containing A. We define $\emptyset^* = \{\bot\}$. $\mathcal{A}ct_\bot(D)$ denotes the cpo $\{\bot\} \cup Act \times D$ ordered by $\bot \sqsubseteq x$ for all x and

$$(\alpha, x) \sqsubseteq (\beta, y) \;\stackrel{\text{def}}{\Longleftrightarrow}\; \alpha = \beta \,\wedge\, x \sqsubseteq y.$$

If D, D' are cpo's and $f : D \to D'$ is strict and continuous then $\wp_{\text{Hoare}}(f) : \wp_{\text{Hoare}}(D) \to \wp_{\text{Hoare}}(D')$ and $\mathcal{A}ct_\bot(f) : \mathcal{A}ct_\bot(D) \to \mathcal{A}ct_\bot(D')$ are given by

$$\wp_{\text{Hoare}}(f)(A) \stackrel{\text{def}}{=} f(A)^*, \quad \mathcal{A}ct_\bot(f)(x) \stackrel{\text{def}}{=} \begin{cases} \bot & : \text{ if } x = \bot, \\ (\alpha, f(y)) & : \text{ if } x = (\alpha, y). \end{cases}$$

$\wp_{\mathrm{Hoare}}(f)$ and $\mathcal{A}ct_\perp(f)$ are strict and continuous. It is easy to see that $\wp_{\mathrm{Hoare}}$ and $\mathcal{A}ct_\perp$ can be viewed as endofunctors on $CONT_\perp$, and that they are locally continuous. Hence, by Lemma 4.3, we can show the following.

Lemma 4.10. The functor $\mathcal{F}_{\mathrm{cont}} \stackrel{\mathrm{def}}{=} \wp_{\mathrm{Hoare}} \circ \mathcal{A}ct_\perp \circ \mathcal{E}_1 : CONT_\perp \to CONT_\perp$ is locally continuous.

By the results of (Abramsky and Jung 1994), $\mathcal{F}_{\mathrm{cont}}$ has an initial fixed point which is simultaneously the initial algebra and final coalgebra of $\mathcal{F}_{\mathrm{cont}}$ as shown by (Rutten and Turi 1993).

Notation 4.11. Let $\mathbf{D}$ denote the initial fixed point of $\mathcal{F}_{\mathrm{cont}}$.

In what follows we deal with the isomorphism as an equality, i.e. if $(\mathbf{D}, j)$ is the initial fixed point of $\mathcal{F}_{\mathrm{cont}}$ then we suppose $\mathbf{D} = \mathcal{F}_{\mathrm{cont}}(\mathbf{D})$ and $j = id_{\mathbf{D}}$. Note that the partial order on $\mathbf{D}$ is the inclusion. The bottom element $\perp_{\mathbf{D}}$ in $\mathbf{D}$ is $\{\perp\}$ where $\perp$ denotes the bottom element in $\mathcal{A}ct_\perp(\mathbf{D})$. If $(x_i)_{i\in I}$ is a directed family of elements in $\mathbf{D}$ then the least upper bound $\bigsqcup x_i$ is $(\bigcup x_i)^*$, the Scott-closure of $\bigcup x_i$ in $\mathcal{A}ct_\perp(\mathbf{D})$. If A, B are finite subsets of $\mathcal{A}ct_\perp(\mathbf{D})$ then $A^* = A\downarrow$ and $A^* \subseteq B^*$ if and only if $A \sqsubseteq_L B$ where $\sqsubseteq_L$ denotes the lower preorder

$$A \sqsubseteq_L B \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad \forall\, x \in A \;\exists\, y \in B : \;\; x \sqsubseteq y.$$

The following is standard. Let $\pi_0^{\mathbf{D}} : \mathbf{D} \to \mathbf{D}$, $\pi_0^{\mathbf{D}}(x) \stackrel{\mathrm{def}}{=} \perp_{\mathbf{D}}$ for all $x \in \mathbf{D}$ and $\pi_{n+1}^{\mathbf{D}} \stackrel{\mathrm{def}}{=} \mathcal{F}_{\mathrm{cont}}\left(\pi_n^{\mathbf{D}}\right)$. Then,

$$\pi_{n+1}^{\mathbf{D}}(x) \;=\; \{(\alpha, \mathcal{E}_1(\pi_n^{\mathbf{D}})(E)) : (\alpha, E) \in x\}^*.$$

$(\pi_n^{\mathbf{D}})_{n\geq 0}$ is a monotone sequence of strict and continuous functions on $\mathbf{D}$ satisfying

— $\pi_n^{\mathbf{D}} \circ \pi_k^{\mathbf{D}} = \pi_k^{\mathbf{D}} \circ \pi_n^{\mathbf{D}} = \pi_n^{\mathbf{D}}$ for all $0 \leq n \leq k$

— $\bigsqcup_{n\geq 0} \pi_n^{\mathbf{D}} \;=\; id_{\mathbf{D}}$

— $x \subseteq y$ iff $\pi_n^{\mathbf{D}}(x) \subseteq \pi_n^{\mathbf{D}}(y)$ for all $n \geq 0$.

If X is a set then we suppose the set of functions $X \to \mathbf{D}$ to be endowed with the partial order $f_1 \sqsubseteq f_2$ iff $f_1(x) \sqsubseteq f_2(x)$ for all $x \in X$. Then, $X \to \mathbf{D}$ is a cpo. We often use the following fact.

Lemma 4.12. Each continuous operator $F : (X \to \mathbf{D}) \to (X \to \mathbf{D})$ with $F(\pi_n^{\mathbf{D}} \circ f) = \pi_{n+1}^{\mathbf{D}} \circ F(f)$ has a unique fixed point.

Proof. The existence of a fixed point f follows by Tarski's fixed point theorem. If f and f' are fixed points of F then it can be shown by induction on n that $\pi_n^{\mathbf{D}} \circ f = \pi_n^{\mathbf{D}} \circ f'$. Hence,

$$f(x) \;=\; \bigsqcup_{n\geq 0} \pi_n^{\mathbf{D}}(f(x)) \;=\; \bigsqcup_{n\geq 0} \pi_n^{\mathbf{D}}(f'(x)) \;=\; f'(x)$$

for all $x \in \mathbf{D}$. □

The following theorem shows the connection between *ProbTree* and $\mathbf{D}$.

Theorem 4.13. There exists a unique function $\imath_{\mathbf{D}} : \mathit{ProbTree} \to \mathbf{D}$ such that $\imath_{\mathbf{D}}(\mathcal{T})$ is the Scott-closure of the set consisting of those pairs (α, E_ν) where $\nu = \mathcal{D}_1(\imath_{\mathbf{D}})(\mu)$ and $\mathcal{T} \stackrel{\alpha}{\to} \mu$. This function $\imath_{\mathbf{D}}$ satisfies:

$$\mathcal{T} \sqsubseteq_{\text{sim}} \mathcal{T}' \iff \imath_{\mathbf{D}}(\mathcal{T}) \subseteq \imath_{\mathbf{D}}(\mathcal{T}')$$

for all $\mathcal{T}$, $\mathcal{T}' \in \mathit{ProbTree}$.

Proof. For simplicity, we write π_n and $\imath$ rather than $\pi_n^{\mathbf{D}}$ and $\imath_{\mathbf{D}}$. Let $F : (\mathit{ProbTree} \to \mathbf{D}) \to (\mathit{ProbTree} \to \mathbf{D})$ be given by

$$F(f)(\mathcal{T}) \stackrel{\text{def}}{=} A_f(\mathcal{T})^* \quad \text{where} \quad A_f(\mathcal{T}) \stackrel{\text{def}}{=} \{(\alpha, E_{\mathcal{D}_1(f)(\mu)}) : \mathcal{T} \stackrel{\alpha}{\to} \mu\}.$$

Note that – since $\mathcal{T}$ is finitely branching – $A_f(\mathcal{T})$ is finite and hence $F(f)(\mathcal{T}) = A_f(\mathcal{T}) \downarrow$. We have to show that F has a unique fixed point $\imath$ and that this function $\imath$ satisfies: $\imath(\mathcal{T}) \subseteq \imath(\mathcal{T}')$ iff $\mathcal{T} \sqsubseteq_{\text{sim}} \mathcal{T}'$.

Claim 1: The operator F is continuous.

Proof of Claim 1: If $f = \bigsqcup f_i$ then we show that for each $\mathcal{T} \in \mathit{ProbTree}$:

1. $A_{f_i}(\mathcal{T}) \sqsubseteq_L A_f(\mathcal{T})$ (which implies $F(f_i)(\mathcal{T}) \subseteq F(f)(\mathcal{T})$)
2. Whenever $y \in \mathbf{D}$ with $A_{f_i}(\mathcal{T}) \subseteq y$ for all $i \in I$ then $A_f(\mathcal{T}) \subseteq y$. This implies that $F(f)(\mathcal{T}) \subseteq y$ for each upper bound y of $(F(f_i)(\mathcal{T}))_{i \in I}$.

Then, from 1. and 2., $\bigsqcup F(f_i)(\mathcal{T}) = F(f)(\mathcal{T})$.

We have $E_{\mathcal{D}_1(f)(\mu)} = \mathcal{E}_1(f)(\mu) = \bigsqcup \mathcal{E}_1(f_i)(\mu) = \bigsqcup E_{\mathcal{D}_1(f_i)(\mu)}$ by Remark 4.2 and Lemma 4.3. Hence:

1. If $(\alpha, E) \in A_{f_i}(\mathcal{T})$ then $\mathcal{T} \stackrel{\alpha}{\to} \mu$ for some distribution μ with $E = E_{\mathcal{D}_1(f_i)(\mu)}$. Then, $E \sqsubseteq E_{\mathcal{D}_1(f)(\mu)}$. Hence, $(\alpha, E) \sqsubseteq (\alpha, E_{\mathcal{D}_1(f)(\mu)}) \in A_f(\mathcal{T})$.
2. If $y \in \mathbf{D}$, $A_{f_i}(\mathcal{T}) \subseteq y$ for all $i \in I$ and $(\alpha, E) \in A_f(\mathcal{T})$ then $E = E_{\mathcal{D}_1(f)(\mu)}$ for some distribution μ where $\mathcal{T} \stackrel{\alpha}{\to} \mu$. Then, $(\alpha, E_{\mathcal{D}_1(f_i)(\mu)}) \in A_{f_i}(\mathcal{T}) \subseteq y$. Since $(\alpha, E) = \bigsqcup (\alpha, E_{\mathcal{D}_1(f_i)(\mu)})$ in $\mathcal{A}ct_\perp(\mathbf{D})$ and y is lub-closed we get $(\alpha, E) \in y$.

Definition: Let $\imath : \mathit{ProbTree} \to \mathbf{D}$ be the least fixed point of F (Tarski's fixed point theorem). Then, $\imath(\mathcal{T}) = F(\imath)(\mathcal{T}) = A_\imath(\mathcal{T}) \downarrow$. If $\mathcal{T} \in \mathit{ProbTree}$ then we put

$$A_n(\mathcal{T}) \stackrel{\text{def}}{=} \{ (\alpha, E_{\mathcal{D}_1(\pi_{n-1} \circ \imath)(\mu)}) : \mathcal{T} \stackrel{\alpha}{\to} \mu\}.$$

Then, $\pi_n(\imath(\mathcal{T})) = A_n(\mathcal{T}) \downarrow$. Thus, $\pi_n(\imath(\mathcal{T})) \subseteq \pi_n(\imath(\mathcal{T}'))$ iff $A_n(\mathcal{T}) \sqsubseteq_L A_n(\mathcal{T}')$.

Claim 2: For all $\mathcal{T}$, $\mathcal{T}' \in \mathit{ProbTree}$, $\imath(\mathcal{T}) \subseteq \imath(\mathcal{T}')$ iff $\mathcal{T} \sqsubseteq_{\text{sim}} \mathcal{T}'$.

Proof of Claim 2: Since all probabilistic trees are finitely branching (and hence image-finite) it is sufficient to show that $\pi_n(\imath(\mathcal{T})) \subseteq \pi_n(\imath(\mathcal{T}'))$ iff $\mathcal{T} \sqsubseteq_n \mathcal{T}'$ (Lemma 2.12).

We prove this by induction on n. In the case $n = 0$ there is nothing to show. In the induction step $n \Longrightarrow n+1$ we suppose $\pi_n(\imath(\mathcal{T})) \subseteq \pi_n(\imath(\mathcal{T}'))$ iff $\mathcal{T} \sqsubseteq_n \mathcal{T}'$ for all $\mathcal{T}$, $\mathcal{T}' \in \mathit{ProbTree}$.

1. Let $\mathcal{T}$, $\mathcal{T}' \in \mathit{ProbTree}$, $\pi_{n+1}(\imath(\mathcal{T})) \subseteq \pi_{n+1}(\imath(\mathcal{T}'))$ and let $\mathcal{T} \stackrel{\alpha}{\to} \mu$ be a transition. Let $\nu = \mathcal{D}_1(\pi_n \circ \imath)(\mu)$. Then,

$$(\alpha, E_\nu) \in A_{n+1}(\mathcal{T}) \sqsubseteq_L A_{n+1}(\mathcal{T}').$$

Hence, there exists $(\alpha, E') \in A_{n+1}(\mathcal{T}')$ with $E_\nu \sqsubseteq E'$. By definition of $A_{n+1}(\mathcal{T}')$ there exists a transition $\mathcal{T}' \xrightarrow{\alpha} \mu'$ with $E' = E_{\nu'}$ and $\nu' = \mathcal{D}_1(\pi_n \circ \imath)(\mu')$. By Theorem 4.5, $\nu \preceq_{\text{sim}} \nu'$. By Remark 2.4, $\mu \preceq_R \nu$, $\mu' \preceq_R \nu'$ where

$$R \stackrel{\text{def}}{=} \{(\mathcal{T}_1, \pi_n(\imath(\mathcal{T}_1)) : \mathcal{T}_1 \in \mathit{ProbTree}\}.$$

Using Remark 2.2 and 2.3 we obtain $\mu \preceq_{R'} \mu'$ where

$$R' \stackrel{\text{def}}{=} R \circ \subseteq \circ R^{-1} = \{ (\mathcal{T}_1, \mathcal{T}_1') : \pi_n(\imath(\mathcal{T}_1)) \subseteq \pi_n(\imath(\mathcal{T}_1')) \}.$$

By induction hypothesis, $R' \subseteq \sqsubseteq_n$. Hence, $\mu \preceq_{\sqsubseteq_n} \mu'$. Thus, $\mathcal{T} \sqsubseteq_n \mathcal{T}'$.

2. Let $\mathcal{T}, \mathcal{T}' \in \mathit{ProbTree}$, $\mathcal{T} \sqsubseteq_{n+1} \mathcal{T}'$. It suffices to show that $A_{n+1}(\mathcal{T}) \sqsubseteq_L A_{n+1}(\mathcal{T}')$. Let $(\alpha, E) \in A_{n+1}(\mathcal{T})$. There exists a transition $\mathcal{T} \xrightarrow{\alpha} \mu$ such that $E = E_\nu$ where $\nu = \mathcal{D}_1(\pi_n \circ \imath)(\mu)$. Since $\mathcal{T} \sqsubseteq_{n+1} \mathcal{T}'$ there exists a transition $\mathcal{T}' \xrightarrow{\alpha} \mu'$ with $\mu \preceq_{\sqsubseteq_n} \mu'$. Then, $(\alpha, E_{\nu'}) \in A_{n+1}(\mathcal{T}')$ where $\nu' = \mathcal{D}_1(\pi_n \circ \imath)(\mu')$. By Remark 2.4, $\mu \preceq_R \nu$, $\mu' \preceq_R \nu'$ where R is as above. Using Remark 2.2 and 2.3 we obtain $\nu \preceq_{R''} \nu'$ where

$$R'' \stackrel{\text{def}}{=} R^{-1} \circ \sqsubseteq_n \circ R = \{(\pi_n(\imath(\mathcal{T}_1)), \pi_n(\imath(\mathcal{T}_1'))) : \mathcal{T}_1 \sqsubseteq_n \mathcal{T}_1'\}.$$

By induction hypothesis we obtain $\nu \preceq_{\text{sim}} \nu'$. By Theorem 4.5, $E_\nu \sqsubseteq E_{\nu'}$. Thus, $(\alpha, E) = (\alpha, E_\nu) \sqsubseteq (\alpha, E_{\nu'}) \in A_{n+1}(\mathcal{T}')$.

Claim 3: If $\imath' : \mathit{ProbTree} \to \mathbf{D}$ is also a fixed point of F then $\imath' = \imath$.

Proof of Claim 3: It is easy to see that $F(\pi_n \circ f) = \pi_{n+1} \circ F(f)$ for all functions $f : \mathit{ProbTree} \to \mathbf{D}$. Hence, by Lemma 4.12, F has a unique fixed point. Therefore, $\imath' = \imath$. □

The final semantics of Section 3.3 yields a semantics on $\mathbf{D}$ which is fully abstract w.r.t. the simulation preorder in the following sense. If $\mathcal{P}$ is a pointed transition system then $\imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}))$ is an element of $\mathbf{D}$ and by Theorem 3.3 and Theorem 4.13:

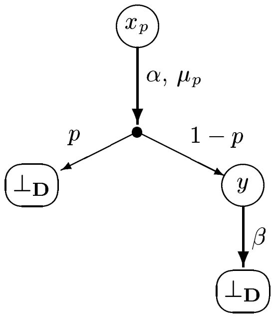
$$\mathcal{P} \sqsubseteq_{\text{sim}} \mathcal{P}' \iff \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P})) \subseteq \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}')).$$

The element $\imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}))$ can be considered as the simulation equivalence class of $\mathcal{P}$.

Example 4.14. We consider the pointed transition systems $\mathcal{P}$ and $\mathcal{P}'$ of Example 3.6 (cf. Figure 2). $\mathcal{P}$ and $\mathcal{P}'$ are represented in $\mathbf{D}$ by

$$\begin{aligned} \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P})) &= \{(\alpha, E_\nu)\}^* &= \{\bot\} \cup \{(\alpha, E) : E \in \mathbf{E}_{\frac{2}{3}}\}, \\ \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}')) &= \{(\alpha, E_{\nu'})\}^* &= \{\bot\} \cup \{(\alpha, E) : E \in \mathbf{E}_{\frac{1}{2}}\}. \end{aligned}$$

Here $\nu = \mu_{\frac{2}{3}}$, $\nu' = \mu_{\frac{1}{2}}$, $\mathbf{E}_p = \{E_{\mu_q} : p \le q \le 1\}$ where μ_p is is the unique distribution on $\mathbf{D}$ with $\mu_p(\bot_{\mathbf{D}}) = p$ and $\mu_p(\{(\beta, \mu^1_{\bot_{\mathbf{D}}})\}^*) = 1 - p$.



The picture above shows the "maximal transitions" of $x_p = \{(\alpha, E_{\mu_p})\}^*$. For each $x \in \mathbf{D}$, each element $(\alpha, E_\mu) \in x$ can be viewed as a "transition" $x \xrightarrow{\alpha} \mu$. Maximality of a transition $x \xrightarrow{\alpha} \mu$ means that, whenever $x \xrightarrow{\alpha} \nu$, then $\nu \preceq_{\mathrm{sim}} \mu$. ■

In the non-probabilistic case, a domain-theoretic model $\mathbf{D}_{\mathrm{nonprob}}$ for the simulation preorder can be obtained from the equation $D \cong \wp_{\mathrm{Hoare}}(\mathcal{A}ct_\perp(D))$ which can be solved in the category of *SFP* domains. The set *Tree* of finitely branching trees (the final solution of $X \cong \wp_{\mathrm{fin}}(\mathcal{A}ct(X))$ in *SET*) can be "embedded" into $\mathbf{D}_{\mathrm{nonprob}}$ via a function $\imath : \mathit{Tree} \to \mathbf{D}_{\mathrm{nonprob}}$ similarly to the way *ProbTree* is "embedded" into $\mathbf{D}$ via $\imath_{\mathbf{D}}$. The images of finitely branching trees of finite height w.r.t. $\imath$ are the compact elements of $\mathbf{D}_{\mathrm{nonprob}}$. In particular, the set $\imath(\mathit{Tree})$ is a basis of $\mathbf{D}_{\mathrm{nonprob}}$. Moreover, if the underlying alphabet *Act* is finite then, for the "n-th projection" $\pi_n : \mathbf{D}_{\mathrm{nonprob}} \to \mathbf{D}_{\mathrm{nonprob}}$, the elements of $\pi_n(\imath(\mathit{Tree}))$ are compact and $\pi_n(\mathbf{D}_{\mathrm{nonprob}}) \subseteq \imath(\mathit{Tree})$. For further details about the non-probabilistic case see (Baier 1997). The situation is different in the probabilistic case:

— The elements of $\pi_n^{\mathbf{D}}(\imath_{\mathbf{D}}(\mathit{ProbTree}))$ are not compact (cf. Remark 4.15).
— $\imath_{\mathbf{D}}(\mathit{ProbTree})$ is not a basis of $\mathbf{D}$ (cf. Remark 4.17).
— If $n \geq 2$ then $\pi_n^{\mathbf{D}}(\mathbf{D}) \not\subseteq \imath_{\mathbf{D}}(\mathit{ProbTree})$ (cf. Remark 4.18).

Remark 4.15. *The elements of $\pi_n^{\mathbf{D}}(\imath_{\mathbf{D}}(\mathit{ProbTree})) \setminus \pi_1^{\mathbf{D}}(\imath_{\mathbf{D}}(\mathit{ProbTree}))$, $n \geq 2$, are not compact.*

Let $n \geq 2$ and $x \in \pi_n^{\mathbf{D}}(\imath_{\mathbf{D}}(\mathit{ProbTree})) \setminus \pi_1^{\mathbf{D}}(\imath_{\mathbf{D}}(\mathit{ProbTree}))$. Then, there exists an element $(\alpha, E_\mu) \in x$ which is maximal in x and with $\mu(y) > 0$ for some $y \in \mathbf{D}$, $y \neq \{\perp\}$. We choose $N \geq 0$ with $1/2^N < \mu(y)$ and for $n \geq N$ we put

$$\mu_n(z) \stackrel{\mathrm{def}}{=} \begin{cases} \mu(z) & : \text{ if } z \neq \{\perp\},\ z \neq y \\ \mu(\{\perp\}) + 1/2^n & : \text{ if } z = \{\perp\} \\ \mu(y) - 1/2^n & : \text{ if } z = y. \end{cases}$$

Then, $\bigsqcup E_{\mu_n} = E_\mu$. Hence, $x = \bigsqcup x_n$ where $x_n \stackrel{\mathrm{def}}{=} A_n^*$, $A_n \stackrel{\mathrm{def}}{=} (x \setminus \{(\alpha, E_\mu)\}) \cup \{(\alpha, E_{\mu_n})\}$ but $x \not\sqsubseteq x_n$ for all $n \geq N$. Thus, x is not compact. ■

Remark 4.16. $\mathcal{D}_1(\mathbf{D})$ *(as a subspace of $\mathcal{E}_1(\mathbf{D})$) is not a basis of $\mathcal{E}_1(\mathbf{D})$.*

We give an example for an evaluation $E \in \mathcal{E}_1(\mathbf{D})$ that cannot be written as $E = \bigsqcup E_\mu$. Let $y \stackrel{\mathrm{def}}{=} \{(\beta, E_\mu)\}^*$ where $\mu = \mu^1_{\perp_{\mathbf{D}}}$. For $p \in [0,1]$, let μ_p be as in Example 4.14, i.e. μ_p is the unique distribution on $\mathbf{D}$ with $\mu_p(y) = p$ and $\mu_p(\perp_{\mathbf{D}}) = 1 - p$. Let

$$x_p \stackrel{\text{def}}{=} \{(\alpha, E_{\mu_p})\}^*, \quad U_p \stackrel{\text{def}}{=} \mathbf{D} \setminus x_{1-p} \downarrow.$$

We define $E \in \mathcal{E}_1(\mathbf{D})$ by $E(U) \stackrel{\text{def}}{=} \sup\{q : x_{1-q} \in U\}$ where $\sup \emptyset \stackrel{\text{def}}{=} 0$. As $x_{1-q} \in U_p$ iff $q < p$ we have $E(U_p) = p$.

Claim 1: If $\mu \in \mathcal{D}_1(\mathbf{D})$, $E_\mu \sqsubseteq E$ then $\mu(x) = 0$ for all $x \in \mathbf{D} \setminus (\{\bot_\mathbf{D}\} \cup \{x_p : p \in [0,1]\})$.

Proof of Claim 1: First we observe that $\mathbf{D} \setminus \{x_p : p \in [0,1]\} \subseteq \mathbf{D} \setminus x_1 \downarrow = U_0$. As U_0 is Scott-open we have $\mu[U_0] \leq E(U_0) = 0$. Hence, $\mu(x) = 0$ for all $x \in U_0$.

We suppose that $E = \bigsqcup_{\mu \in \mathcal{M}} E_\mu$ where $\mathcal{M} \subseteq \mathcal{D}_1(\mathbf{D})$ such that $\{E_\mu : \mu \in \mathcal{M}\}$ is directed. (I.e. $\mathcal{M}$ is directed w.r.t. $\preceq_{\text{sim}}$.) Then, for all $\mu \in \mathcal{M}$ and $p \in [0,1]$,

$$p \;=\; \sup\{\mu[U_p] : \mu \in \mathcal{M}\}.$$

Claim 2: For all $\varepsilon > 0$ there is some $\mu \in \mathcal{M}$ such that $\mu[U_p] \geq p - \varepsilon$ for all $p \in [0,1]$.

Proof of Claim 2: For each $p \in [0,1]$ we choose some μ_p with $\mu_p[U_p] \geq p - \varepsilon$ (axiom of choice). Let X_p be a finite subset of $\mathbf{D}$ such that $\mu_p[\mathbf{D} \setminus X_p] < \varepsilon/2$. There exists some $\delta_p > 0$ with $\delta_p < \varepsilon/2$ and $\{x_q : q \in \,]p - \delta_p, p + \delta_p[\,\} \cap X_p \subseteq \{p\}$. As $[0,1]$ is compact there exists $p_1, \ldots, p_k \in [0,1]$ such that

$$[0,1] \;\subseteq\; \bigcup_{i=1}^{k} \,]p_i - \delta_i, p_i + \delta_i[.$$

where $\delta_i = \delta_{p_i}$. For all $q \in]p_i - \delta_i, p_i + \delta_i[$ and $i = 1, \ldots, k$,

$$\mu_{p_i}[U_q] \;=\; \mu_{p_i}[U_{p_i}] \;\geq\; p_i - \frac{1}{2} \cdot \varepsilon \;\geq\; q - \varepsilon.$$

We choose some $\mu \in \mathcal{M}$ with $\mu_{p_i} \preceq_{\text{sim}} \mu$. Then, $\mu[U_q] \geq q - \varepsilon$ for all $q \in [0,1]$.

Let (μ_n) be a sequence in $\mathcal{M}$ such that $\mu_1 \preceq_{\text{sim}} \mu_2 \preceq_{\text{sim}} \ldots$ and $\mu_n[U_p] \geq p - 1/2^n$ for all $n \geq 1$ and $p \in [0,1]$. (The existence of such a sequence follows by Claim 2 and the fact that $\mathcal{M}$ is directed w.r.t. $\preceq_{\text{sim}}$.)

Claim 3: $\lim_{n \to \infty} \mu_n(x_{1-p}) = 0$ for all $p \in [0,1]$.

Proof of Claim 3: Let $\varepsilon > 0$ and $n \geq 1$ such that $1/2^{n-1} < \varepsilon$. For all $q \in [0,1]$, $q > p$, we have $x_{1-p} \in U_q \setminus U_p$ and $U_p \subseteq U_q$. Hence, for all q with $p < q < p + 1/2^n$,

$$q \;\geq\; \mu_n[U_q] \;\geq\; \mu_n[U_p] + \mu_n(x_{1-p}) \;\geq\; p - \frac{1}{2^n} + \mu_n(x_{1-p}) \;\geq\; q - \frac{1}{2^{n-1}} + \mu_n(x_{1-p}).$$

Thus, $\mu_n(x_{1-p}) \leq 1/2^{n-1} < \varepsilon$.

Claim 4: $\lim_{n \to \infty} \mu_n(\bot_\mathbf{D}) = 0$.

Proof of Claim 4: We suppose that there is some $\varepsilon > 0$ with $\mu_n(\bot_\mathbf{D}) \geq \varepsilon$ for infinitely many n. Let (μ_{n_k}) be a subsequence of (μ_n) with $\mu_{n_k}(\bot_\mathbf{D}) \geq 0$ for all $k \geq 1$. We choose some $k \geq 1$ where $1/2^{n_k} < \varepsilon$. Since $\bot_\mathbf{D} \notin U_1$ we get

$$1 - \frac{1}{2^{n_k}} \;\leq\; \mu_{n_k}[U_1] \;\leq\; 1 - \mu_{n_k}(\bot_\mathbf{D}) \;\leq 1 - \varepsilon.$$

Contradiction.

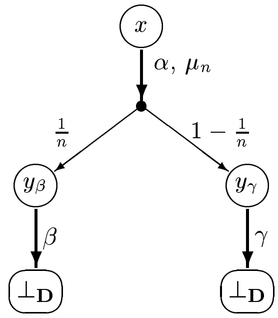
Let $X \stackrel{\text{def}}{=} \bigcup_{n \geq 1} \{x \in D : \mu_n(x) > 0\}$. By Claim 1, $X \subseteq \{\bot_\mathbf{D}\} \cup \{x_p : p \in [0,1]\}$. Thus,

$$1 \;=\; \lim_{n \to \infty} \sum_{x \in X} \mu_n(x) \;=\; \sum_{x \in X} \lim_{n \to \infty} \mu_n(x) \;=\; 0$$

by Claim 3 and 4. Contradiction. ■

Remark 4.17. *$\imath_{\mathbf{D}}(ProbTree)$ is not a basis of $\mathbf{D}$.*
We consider the element $y = \{(\alpha, E)\}^*$ of $\mathbf{D}$ where E is as in Remark 4.16. We suppose that y can be written in the form $y = \bigsqcup X$ for some directed subset X of $\imath_{\mathbf{D}}(ProbTree)$. Then, y is the Scott closure of $\bigcup_{x \in X} x$ in $\mathcal{A}ct_{\perp}(\mathbf{D})$. Since $X \subseteq \imath_{\mathbf{D}}(ProbTree)$ each element x of X is of the form $\{(\alpha, E_{\nu_1^x}), \ldots, (\alpha, E_{\nu_{r_x}^x})\}^*$ where ν_i^x are distributions with $E_{\nu_i^x} \sqsubseteq E$. Let $\mathcal{M} = \{\nu_i^x : i = 1, \ldots, r_x, x \in X\}$. It is easy to see that $\mathcal{M}$ is directed and $E = \bigsqcup\{E_\mu : \mu \in \mathcal{M}\}$, which is impossible as shown in Remark 4.16. ■

Remark 4.18. *If $n \geq 2$ then $\pi_n^{\mathbf{D}}(\mathbf{D}) \not\subseteq \imath_{\mathbf{D}}(ProbTree)$.*[§]
For instance, consider $x \stackrel{\text{def}}{=} A^*$ where $A \stackrel{\text{def}}{=} \{(\alpha, E_{\mu_n}) : n \geq 1\}$ and μ_n is the unique distribution on $\mathbf{D}$ with $\mu_n(y_\beta) = 1/n$ and $\mu_n(y_\gamma) = 1 - 1/n$.



Here, $y_\beta \stackrel{\text{def}}{=} \{(\beta, E_{\mu^1_{\perp_D}})\}^*$, $y_\gamma \stackrel{\text{def}}{=} \{(\gamma, E_{\mu^1_{\perp_D}})\}^*$. The elements (α, E_{μ_n}), $n \geq 1$ are pairwise incomparable. Hence, x cannot be written in the form $x = X^* (= X\downarrow)$ where X is finite. Therefore, $x \notin \imath_{\mathbf{D}}(ProbTree)$, but $x = \pi_2^{\mathbf{D}}(x) \in \pi_2^{\mathbf{D}}(\mathbf{D})$. ■

4.3. *A metric domain equation for probabilistic trees*

This section shows that the equation $X \cong \wp_{\text{comp}}(Act \times \mathcal{E}_1(M))$ can be solved in the category of complete ultrametric spaces where the fixed point theorem of (Rutten and Turi 1993), which originates from the results of (America and Rutten 1989), is used.

If M is a complete ultrametric space then $\wp_{\text{comp}}(M)$ denotes the collection of compact subsets of M. We suppose $\wp_{\text{comp}}(M)$ to be equipped with the Hausdorff metric:

$$d(X, Y) \stackrel{\text{def}}{=} \max\left\{ \sup_{x \in X} d(x, Y),\ \sup_{y \in Y} d(y, X) \right\}$$

where $d(w, Z) \stackrel{\text{def}}{=} \inf_{z \in Z} d(w, z)$. Then, $\wp_{\text{comp}}(M)$ is a complete ultrametric space, see (Kuratowski 1956). If $f : M \to M'$ is a non-expansive function then $\wp_{\text{comp}}(f)$:

[§] Note that $\pi_1^{\mathbf{D}}(\mathbf{D}) = \{\perp_{\mathbf{D}}\} \cup \{(\alpha, E_\mu) : \alpha \in Act\} \subseteq \imath_{\mathbf{D}}(ProbTree)$ where $\mu = \mu^1_{\perp_{\mathbf{D}}}$.

$\wp_{\mathrm{comp}}(M) \to \wp_{\mathrm{comp}}(M')$, $\wp_{\mathrm{comp}}(f)(X) \stackrel{\mathrm{def}}{=} f(X)$ is non-expansive. Hence, we get a functor $\wp_{\mathrm{comp}} : \mathit{CUM} \to \mathit{CUM}$ which is locally non-expansive, see (Rutten and Turi 1993). In order to obtain a locally contracting functor $\mathit{CUM} \to \mathit{CUM}$ we deal with the functor $\mathcal{A}ct_{\frac{1}{2}} : \mathit{CUM} \to \mathit{CUM}$ which is defined as follows. If M is a complete ultrametric space then $\mathcal{A}ct_{\frac{1}{2}}(M)$ is $Act \times M$ endowed with the distance

$$d((\alpha, x),\ (\beta, y)\ ) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & : \text{ if } \alpha \neq \beta \\ \frac{1}{2} \cdot d(x, y) & : \text{ if } \alpha = \beta. \end{cases}$$

If $f : M \to M'$ is a non-expansive function between complete ultrametric spaces then we define $\mathcal{A}ct_{\frac{1}{2}}(f)$ to be the function $\mathcal{A}ct(f)$. It is easy to see that for each complete ultrametric space M and non-expansive function $f : M \to M'$, $\mathcal{A}ct_{\frac{1}{2}}(M)$ is a complete ultrametric space and $\mathcal{A}ct_{\frac{1}{2}}(f)$ is non-expansive. From Lemma 4.8 we obtain the following.

Lemma 4.19. The functor $\mathcal{F}_{\mathrm{cum}} \stackrel{\mathrm{def}}{=} \wp_{\mathrm{comp}} \circ \mathcal{A}ct_{\frac{1}{2}} \circ \mathcal{E}_1 : \mathit{CUM} \to \mathit{CUM}$ is locally contracting in the sense of (Rutten and Turi 1993).

By the results of (Rutten and Turi 1993), $\mathcal{F}_{\mathrm{cum}}$ has a unique fixed point which is at the same time the final coalgebra and initial algebra. Note that (Rutten and Turi 1993) deal with the category *CMS* of complete metric spaces instead of the subcategory *CUM*. It is easy to see that the fixed point theorem of (Rutten and Turi 1993), and also those of (America and Rutten 1989), carry over to the category *CUM*.

Notation 4.20. Let $\mathbf{M}$ denote the unique fixed point of $\mathcal{F}_{\mathrm{cum}}$.

In what follows, we deal with the isomorphism as an equality, i.e. if $(\mathbf{M}, j)$ is the unique fixed point of $\mathcal{F}_{\mathrm{cum}}$ then we suppose $\mathbf{M} = \mathcal{F}_{\mathrm{cum}}(\mathbf{M}))$ and $j = id_{\mathbf{M}}$. The following is standard. Let $\pi_0^{\mathbf{M}} : \mathbf{M} \to \mathbf{M}$, $\pi_0^{\mathbf{M}}(x) \stackrel{\mathrm{def}}{=} \emptyset$ for all $x \in \mathbf{M}$ and $\pi_{n+1}^{\mathbf{M}} \stackrel{\mathrm{def}}{=} \mathcal{F}_{\mathrm{cum}}\left(\pi_n^{\mathbf{M}}\right)$. Then,

$$\pi_{n+1}^{\mathbf{M}}(x) \;=\; \{(\alpha, \mathcal{E}_1(\pi_n^{\mathbf{M}})(E)) : (\alpha, E) \in x\}.$$

We have $\pi_n^{\mathbf{M}} \circ \pi_k^{\mathbf{M}} = \pi_k^{\mathbf{M}} \circ \pi_n^{\mathbf{M}} = \pi_n^{\mathbf{M}}$ for all $0 \leq n \leq k$ and $x = \lim_{n \to \infty} \pi_n^{\mathbf{M}}(x)$ for all $x \in \mathbf{M}$. The distance on $\mathbf{M}$ is given by

$$d(x, y) \;=\; \inf \left\{ \frac{1}{2^n} \;:\; \pi_n^{\mathbf{M}}(x) = \pi_n^{\mathbf{M}}(y) \right\}.$$

A function $f : \mathbf{M} \to \mathbf{M}$ is non-expansive (resp. contracting) if and only if $\pi_n^{\mathbf{M}} \circ f = \pi_n^{\mathbf{M}} \circ f \circ \pi_n^{\mathbf{M}}$ (resp. $\pi_n^{\mathbf{M}} \circ f = \pi_n^{\mathbf{M}} \circ f \circ \pi_{n-1}$) for all $n \geq 1$. In particular, every function $f : \mathbf{M} \to \mathbf{M}$ with $\pi_n^{\mathbf{M}} \circ f = f \circ \pi_{n-1}^{\mathbf{M}}$ for all $n \geq 1$ is contracting. If X is a set then the set of functions $X \to \mathbf{M}$ is supposed to be equipped with the distance $d(f_1, f_2) \stackrel{\mathrm{def}}{=} \sup_{x \in X} d(f_1(x), f_2(x))$. Then, $X \to \mathbf{M}$ is a complete ultrametric space. We often use the following fact.

Lemma 4.21. Every operator $F : (X \to \mathbf{M}) \to (X \to \mathbf{M})$ with $F\left(\pi_n^{\mathbf{M}} \circ f\right) = \pi_{n+1}^{\mathbf{M}} \circ F(f)$ is contracting and hence has a unique fixed point.

Proof. It is easy to see that F is contracting and hence has a unique fixed point (Banach's fixed point theorem). $\square$

Lemma 4.22. For all $n \geq 1$ and $x \in \mathbf{M}$, $\pi_n^{\mathbf{M}}(x)$ is a finite set consisting of pairs (α, E_μ) where $\alpha \in Act$ and $\mu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$.

Here, we identify each distribution μ on a subspace X of $\mathbf{M}$ with the distribution $x \mapsto \mu(x)$ if $x \in X$, $x \mapsto 0$ if $x \in \mathbf{M} \setminus X$, on $\mathbf{M}$.

Proof. For simplicity, $\pi_n = \pi_n^{\mathbf{M}}$.
Claim 1: If $E \in \mathcal{E}_1(\mathbf{M})$ then $\mathcal{E}_1(\pi_{n-1})(E) = E_\mu$ for some distribution $\mu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$.
Proof of Claim 1: $\mathbf{M}$ can be written as disjoint union of the open balls $B(x, 1/2^{n-2})$, $x \in \pi_{n-1}(\mathbf{M})$. In the appendix (Section A.2, Lemma A.10) it is shown that there exists a countable subset N of $\pi_{n-1}(\mathbf{M})$ with $E(\ B(x, 1/2^{n-2})\ ) \neq 0$ implies $x \in N$. For all opens U,

$$\mathcal{E}_1(\pi_{n-1})(E)(U) \ = \ E(\pi_{n-1}^{-1}(U)) \ = \ \sum_{x \in N \cap U} E(\ B(x, 1/2^{n-2})\ ) \ = \ \mu[U]$$

where $\mu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$ is given by $\mu(x) \stackrel{\text{def}}{=} 0$ if $x \notin N$ and $\mu(x) \stackrel{\text{def}}{=} E(B(x, 1/2^{n-2}))$ if $x \in N$.
Claim 2: If E, $E' \in \mathcal{E}_1(\mathbf{M})$ and $d(\ \mathcal{E}_1(\pi_{n-1})(E),\ \mathcal{E}_1(\pi_{n-1})(E')\ ) \ \leq \ 1/2^{n-1}$ then

$$\mathcal{E}_1(\pi_{n-1})(E) \ = \ \mathcal{E}_1(\pi_{n-1})(E').$$

Proof of Claim 2: Because of Claim 1 it suffices to show that $d(E_\mu, E_{\mu'}) \ \leq \ 1/2^{n-1}$ implies $\mu = \mu'$ where μ, $\mu' \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$. Let $x \in \pi_{n-1}(\mathbf{M})$. Then, $B(x, 1/2^{n-2}) \cap \pi_{n-1}(\mathbf{M}) \ = \{x\}$. Hence,

$$\mu(x) \ = \ \mu[B(x, 1/2^{n-2})] \ = \ \mu'[B(x, 1/2^{n-2})] \ = \ \mu'(x)$$

for all $x \in \pi_{n-1}(\mathbf{M})$. Therefore, $\mu = \mu'$.
Claim 3: $\pi_n(x)$ is a finite set consisting of pairs (α, E_μ) where $\mu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$.
Proof of Claim 3: The elements of $\pi_n(x)$ are of the form (α, E_μ) where $\mu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M}))$ (see Claim 1). Since $\pi_n(x)$ is compact (as a subset of $\mathcal{A}ct_{\frac{1}{2}}(\mathcal{E}_1(\mathbf{M}))$) and since $\pi_n(x) \ \subseteq \bigcup_{\xi \in \pi_n(x)} \ B(\xi, 1/2^n)$¶ there exists a finite subset Ξ of $\pi_n(x)$ with

$$\pi_n(x) \ \subseteq \ \bigcup_{\xi \in \Xi} \ B(\xi, 1/2^n).$$

We show that $\pi_n(x) = \Xi$. Let $(\alpha, E_\mu) \in \pi_n(x)$. There exists $\xi \in \Xi$ with $d(\xi, (\alpha, E_\mu)) < 1/2^n$. ξ is of the form (β, E_ν) where $\nu \in \mathcal{D}_1(\pi_{n-1}(\mathbf{M})))$. Then, $\alpha = \beta$ and $d(E_\mu, E_\nu) < 1/2^{n-1}$. By Claim 2, it follows that $\mu = \nu$, and therefore $\xi = (\alpha, E_\mu) \in \Xi$. Hence, $\pi_n(x) = \Xi$ is finite. $\square$

Theorem 4.23. *ProbTree* is a dense subspace of $\mathbf{M}$. More precisely, there exists a unique

¶ Note that $B(\xi, \rho)$ is an open ball in $\mathcal{A}ct_{\frac{1}{2}}(\mathcal{E}_1(\mathbf{M}))$.

function $\imath_{\mathbf{M}} : \mathit{ProbTree} \to \mathbf{M}$ such that for all probabilistic trees $\mathcal{T}$,

$$\imath_{\mathbf{M}}(\mathcal{T}) \;=\; \{(\alpha, E_{\mathcal{D}_1(\imath_{\mathbf{M}})(\mu)}) : \mathcal{T} \xrightarrow{\alpha} \mu\}.$$

This function $\imath_{\mathbf{M}}$ is injective and $\imath_{\mathbf{M}}(\mathit{ProbTree})$ is a dense subspace of $\mathbf{M}$.

Proof. We write π_n and $\imath$ instead of $\pi_n^{\mathbf{M}}$ and $\imath_{\mathbf{M}}$. Let $F : (\mathit{ProbTree} \to \mathbf{M}) \to (\mathit{ProbTree} \to \mathbf{M})$ be given by

$$F(f)(\mathcal{T}) \;\stackrel{\text{def}}{=}\; \{\ (\alpha, E_{\mathcal{D}_1(f)(\mu)}) : \mathcal{T} \xrightarrow{\alpha} \mu\}.$$

Note that – since $\mathcal{T}$ is finitely branching – $F(f)(\mathcal{T})$ is finite and hence compact. We have to show that F has a unique fixed point $\imath$ and that this function $\imath$ is injective and $\imath(\mathit{ProbTree})$ a dense subspace of $\mathbf{M}$. It is easy to see that $F(\pi_n \circ f) \;=\; \pi_{n+1} \circ F(f)$ for all functions $f : \mathit{ProbTree} \to \mathbf{M}$. Hence, F is contracting (Lemma 4.21).

Definition: Let $\imath$ be the unique fixed point of F (Banach's fixed point theorem).

Claim 1: $\imath$ is injective.

Proof of Claim 1: Because of Lemma 2.6 it suffices to show that $\imath(\mathcal{T}) = \imath(\mathcal{T}')$ implies $\mathcal{T} \sim_n \mathcal{T}'$ for all $n \geq 0$. (Since $\mathcal{T}$, $\mathcal{T}'$ are finitely branching (and therefore image-finite) we get $\mathcal{T} \sim \mathcal{T}'$. Hence, $\mathcal{T} = \mathcal{T}'$ (Corollary 3.4).)

We show by induction on n that $\pi_n(\imath(\mathcal{T})) = \pi_n(\imath(\mathcal{T}'))$ iff $\mathcal{T} \sim_n \mathcal{T}'$. In the basis of induction ($n = 0$) there is nothing to show. In the induction step $n \Longrightarrow n+1$ we suppose that, for all $\mathcal{T}_1$, $\mathcal{T}_1' \in \mathit{ProbTree}$, $\pi_n(\imath(\mathcal{T}_1)) = \pi_n(\imath(\mathcal{T}_1'))$ iff $\mathcal{T}_1 \sim_n \mathcal{T}_1'$.

1. Let $\pi_{n+1}(\imath(\mathcal{T})) = \pi_{n+1}(\imath(\mathcal{T}'))$ and $\mathcal{T} \xrightarrow{\alpha} \mu$. Then, $(\alpha, E_\nu) \in \pi_{n+1}(\imath(\mathcal{T}))$ where $\nu \;=\; \mathcal{D}_1(\pi_n \circ \imath)(\mu)$. Since $(\alpha, E_\nu) \in \pi_{n+1}(\imath(\mathcal{T}'))$ there exists a transition $\mathcal{T}' \xrightarrow{\alpha} \mu'$ with $\nu \;=\; \mathcal{D}_1(\pi_n \circ \imath)(\mu')$. Let $A \in \mathit{ProbTree}/\sim_n$, $\mathcal{T}_1 \in A$ and $x = \pi_n(\imath(\mathcal{T}_1))$. By induction hypothesis,

   $$A \;=\; \{\mathcal{T}_1' \in \mathit{ProbTree} : x = \pi_n(\imath(\mathcal{T}_1'))\}.$$

   Hence, $A \;=\; \imath^{-1}(\pi_n^{-1}(x))$. Thus, $\mu[A] = \nu(x) = \mu'[A]$. By symmetry we get $\mathcal{T} \sim_{n+1} \mathcal{T}'$.
2. Let $\mathcal{T} \sim_{n+1} \mathcal{T}'$. By symmetry it suffices to show that $\pi_{n+1}(\imath(\mathcal{T})) \subseteq \pi_{n+1}(\imath(\mathcal{T}'))$. Let $(\alpha, E) \in \pi_{n+1}(\imath(\mathcal{T}))$. There is a transition $\mathcal{T} \xrightarrow{\alpha} \mu$ with $E = E_{\mathcal{D}_1(\pi_n \circ \imath)(\mu)}$. Since $\mathcal{T} \sim_{n+1} \mathcal{T}'$ there exists a transition $\mathcal{T}' \xrightarrow{\alpha} \mu'$ with $\mu[A] = \mu'[A]$ for all $A \in \mathit{ProbTree}/\sim_n$. We show $\mathcal{D}_1(\pi_n \circ \imath)(\mu) = \mathcal{D}_1(\pi_n \circ \imath)(\mu')$.
   Let $x \in \mathbf{M}$ and $A \stackrel{\text{def}}{=} \imath^{-1}(\pi_n^{-1}(x))$. By induction hypothesis, $A \in \mathit{ProbTree}/\sim_n$. Thus, $\mathcal{D}_1(\pi_n \circ \imath)(\mu)(x) = \mu[A] = \mu'[A] = \mathcal{D}_1(\pi_n \circ \imath)(\mu')(x)$.

Claim 2: $\imath(\mathit{ProbTree})$ is a dense subspace of $\mathbf{M}$.

Proof of Claim 2: Since $x = \lim \pi_n(x)$ for all $x \in \mathbf{M}$ the set $\bigcup \pi_n(M)$ is a dense subspace of $\mathbf{M}$. Hence, it suffices to show that $\pi_n(\mathbf{M}) \subseteq \imath(\mathit{ProbTree})$ for all $n \geq 0$. We use induction on n. The case $n = 0$ is clear. In the induction step $n \Longrightarrow n+1$ we suppose $\pi_n(\mathbf{M}) \subseteq \imath(\mathit{ProbTree})$. Since $\imath$ is injective there exists a unique function $\jmath : \mathbf{M} \to \mathit{ProbTree}$ such that $\jmath \circ \imath = id_{\mathit{ProbTree}}$. Let $x \in \pi_{n+1}(\mathbf{M})$. By Lemma 4.22, x is of the form

$$x \;=\; \{(\alpha_i, E_{\mu_i}) : i = 1, \ldots, k\}$$

where $\mu_i \in \mathcal{D}_1(\pi_n(\mathbf{M})) \subseteq \mathcal{D}_1(\imath(\mathit{ProbTree}))$. Hence, $\mathcal{D}_1(\imath \circ \jmath)(\mu_i) = \mu_i$. Let

$$X \stackrel{\text{def}}{=} \{(\alpha_i, E_{\mathcal{D}_1(\jmath)(\mu_i)}) : i = 1, \ldots, k\}, \quad \mathcal{T} \stackrel{\text{def}}{=} e^{-1}(X)$$

where $e : \mathit{ProbTree} \to \wp_{\text{fin}}(\mathit{Act} \times \mathcal{D}_1(\mathit{ProbTree}))$ is the bijection such that $(\mathit{ProbTree}, e)$ is the final coalgebra of $\wp_{\text{fin}} \circ \mathcal{A}ct \circ \mathcal{D}_1$ (Theorem 3.5). By Remark 4.2,

$$\mathcal{E}_1(\imath)(E_{\mathcal{D}_1(\jmath)(\mu_i)}) = E_{\mathcal{D}_1(\imath\circ\jmath)(\mu_i)} = E_{\mu_i}, \quad i = 1, \ldots, k.$$

Hence, $\imath(\mathcal{T}) = \{(\alpha, \mathcal{E}_1(\imath)(E)) : (\alpha, E) \in X\} = \{(\alpha_i, E_{\mu_i}) : i = 1, \ldots, k\} = x$. Thus, $x \in \imath(\mathit{ProbTree})$. □

By Theorem 3.3 and Theorem 4.23, $\mathbf{M}$ is fully abstract w.r.t. bisimulation. I.e.

$$\mathcal{P} \sim \mathcal{P}' \text{ iff } \imath_{\mathbf{M}}(\mathit{probtree}(\mathcal{P})) = \imath_{\mathbf{M}}(\mathit{probtree}(\mathcal{P}'))$$

for all pointed transition systems $\mathcal{P}$ and $\mathcal{P}'$.

Remark 4.24. The density of $\imath_{\mathbf{M}}(\mathit{ProbTree})$ in $\mathbf{M}$ stands in contrast to Remark 4.17 which shows that $\imath_{\mathbf{D}}(\mathit{ProbTree})$ is not a basis of $\mathbf{D}$. In the proof of Theorem 4.23 (Claim 2) we saw that $\pi_n^{\mathbf{M}}(\mathbf{M}) \subseteq \imath_{\mathbf{M}}(\mathit{ProbTree})$. This should be contrasted with the domain-theoretic setting where $\pi_n^{\mathbf{D}}(\mathbf{D}) \not\subseteq \imath_{\mathbf{D}}(\mathit{ProbTree})$ (cf. Remark 4.18). ■

5. Semantics for Probabilistic *CCS*

In this section we consider a probabilistic extension of Milner's *CCS* (Milner 1989) which is based on the calculi of (Yi and Larsen 1992; Yi 1994; Hansson and Jonsson 1990) that allow for non-deterministic and probabilistic choice. In this respect, our calculus differs from those introduced by (Tofts 1994) or (Giacalone *et al.* 1990), which are probabilistic extensions of *SCCS* (synchronous *CCS*) (Milner 1983) and replace the concept of non-determinism by a probabilistic choice operator. In our calculus *PCCS* (probabilistic *CCS*), the usual prefixing operator $s \mapsto \alpha.s$ is replaced with the (guarded) *probabilistic choice* operator

$$\sum_{i\in I} \alpha_{p_i}.s_i$$

where p_i is a real number between 0 and 1 denoting the probability that after performing α the above process becomes s_i. The syntax of the probabilistic operator derives from Milner's notion of an infinite summation $\sum_v \alpha_v.s_v$, where α_v denotes the input action which transmits the value v to a program variable x, and the behaviour of the continuation process might be dependent on the value v. The difference between our operator and Milner's infinite sum is that we suppose that there exists a distribution on the possible values v; p_i is then the sum of the probabilities for input values v for which the continuation process s_v equals s_i.

Instead of Milner's process equations, for convenience we use declarations to model recursive behaviour. In what follows, *Var* is a set of (process) variables and *Act* a set of atomic actions which contains an internal action τ (representing internal computations of a process which are not visible for the environment) and which is equipped with a function $\mathit{Act} \to \mathit{Act}$, $\alpha \mapsto \overline{\alpha}$, where $\overline{\tau} = \tau$ and $\overline{\overline{\alpha}} = \alpha$. $\overline{\alpha}$ is called the complementary action of α. Synchronization of processes is only possible by performing complementary actions α and $\overline{\alpha}$, $\alpha \neq \tau$. The result of a synchronization is the internal action τ.

The syntax of *PCCS* (Probabilistic *CCS*) is as follows. *Statements* are built from the production system

$$s \quad ::= \quad nil \mid x \mid \sum_{i \in I} \alpha_{p_i}.s_i \mid s_1 \oplus s_2 \mid s_1 \parallel s_2 \mid s[\lambda] \mid s \setminus L$$

where $\alpha \in Act$, $x \in Var$, L is a subset of $Act \setminus \{\tau\}$ with $\overline{L} = L$ (where $\overline{L} = \{\overline{\alpha} : \alpha \in L\}$), $\lambda : Act \to Act$ is a relabelling function (i.e. $\lambda(\overline{\alpha}) = \overline{\lambda(\alpha)}$ and $\lambda(\tau) = \tau$) and I is a nonempty countable indexing set and $(p_i)_{i \in I}$ a family of real numbers $p_i \in [0,1]$ such that $\sum_{i \in I} p_i = 1$. For finite indexing set $I = \{i_1, \ldots, i_n\}$ we also write $\alpha_{p_{i_1}}.s_{i_1} + \ldots + \alpha_{p_{i_n}}.s_{i_n}$ instead of $\sum_{i \in I} \alpha_{p_i}.s_i$. *Guarded statements* are built from the production system

$$g \quad ::= \quad nil \mid \sum_{i \in I} \alpha_{p_i}.s_i \mid g_1 \oplus g_2 \mid g_1 \parallel g_2 \mid g[\lambda] \mid g \setminus L$$

Stmt denotes the set of all statements. A *declaration* is a function $\sigma : Var \to Stmt$. A declaration σ is called guarded iff $\sigma(x)$ is guarded for all $x \in Var$. A *program* is a pair $\mathcal{P} = < \sigma, s >$ consisting of a declaration σ and a statement s. *PCCS* denotes the set of all programs, and *GPCCS* the subset of guarded programs, i.e. all programs $< \sigma, s >$ where σ is a guarded declaration.

The intended meaning of programs $\mathcal{P} = < \sigma, s >$ is that the behaviour of $\mathcal{P}$ is given by the statement s where each occurrence of a variable x in s is considered as a recursive call of the procedure $\sigma(x)$. In other words, a declaration σ corresponds to the family of process equations $\sigma(x) = x$, $x \in Var$. *nil* stands for a process which does not perform any action. $\oplus$ models nondeterministic choice (denoted by $+$ in *CCS*): $s_1 \oplus s_2$ either behaves like s_1 or like s_2. $\parallel$ denotes the usual parallel composition. The operators $s \mapsto s \setminus L$, $s \mapsto s[\lambda]$ model restriction and relabelling: $s \setminus L$ behaves like s as long as s does not perform an action $\alpha \in L$. $s[\lambda]$ behaves like s where each action $\alpha \in Act$ is replaced by $\lambda(\alpha)$.

Our language *PCCS* is contained in the calculus considered by Yi & Larsen (Yi and Larsen 1992) (and the extended version of (Yi 1994) which adds aspects of time and uses a generalized probabilistic choice operator that prescribes intervals of probabilities rather than the exact probabilities) which allows probabilistic choice between arbitrary processes. For instance, the calculus of (Yi and Larsen 1992) allows statements such as $\alpha_{\frac{1}{3}}.nil + \beta_{\frac{2}{3}}.nil$ which denotes a process that offers α with probability $1/3$ and β with probability $2/3$ and terminates after performing α or β respectively. In our calculus, where probabilistic choices are bound by action occurrences, the process above has to be replaced by the process $\tau_{\frac{1}{3}}.\alpha_1.nil + \tau_{\frac{2}{3}}.\beta_1.nil$ which resolves the probabilisic choice between α and β via an internal action τ. The operational semantics of (Yi and Larsen 1992) differs from ours as it distinguishes between action and probability labelled transitions. In (Hansson and Jonsson 1990) a version of *CCS* with time and probability is proposed. The probabilistic aspects of the language considered in (Hansson and Jonsson 1990)

1. $\sum_{i\in I} \alpha_{p_i}.s_i \xrightarrow{\alpha}_\sigma \mu$ where $\mu(s) = \sum_{\substack{i\in I\\ s_i=s}} p_i$
2. $s_1 \oplus s_2 \xrightarrow{\alpha}_\sigma \mu$ if $s_1 \xrightarrow{\alpha}_\sigma \mu$ or $s_2 \xrightarrow{\alpha}_\sigma \mu$
3. $s_1 \parallel s_2 \xrightarrow{\alpha}_\sigma \mu$ if one of the following three conditions is satisfied:
   (i) $s_1 \xrightarrow{\alpha}_\sigma \mu_1$ and $\mu(s) = \begin{cases} \mu_1(s_1') & : \text{ if } s = s_1' \parallel s_2 \\ 0 & : \text{ otherwise} \end{cases}$
   (ii) $s_2 \xrightarrow{\alpha}_\sigma \mu_2$ and $\mu(s) = \begin{cases} \mu_2(s_2') & : \text{ if } s = s_1 \parallel s_2' \\ 0 & : \text{ otherwise} \end{cases}$
   (iii) $\alpha = \tau$ and there exists $\beta \in Act \setminus \{\tau\}$ with $s_1 \xrightarrow{\beta}_\sigma \mu_1$ and $s_2 \xrightarrow{\overline{\beta}}_\sigma \mu_2$ such that
   $$\mu(s) = \begin{cases} \mu_1(s_1') \cdot \mu_2(s_2') & : \text{ if } s = s_1' \parallel s_2' \\ 0 & : \text{ otherwise} \end{cases}$$
4. $s[\lambda] \xrightarrow{\alpha}_\sigma \mu$ if $s \xrightarrow{\beta}_\sigma \mu'$, $\lambda(\beta) = \alpha$ and $\mu(s) = \begin{cases} \mu'(s') & : \text{ if } s = s'[\lambda] \\ 0 & : \text{ otherwise} \end{cases}$
5. $s \setminus L \xrightarrow{\alpha}_\sigma \mu$ if $s \xrightarrow{\alpha}_\sigma \mu'$, $\alpha \notin L$ and $\mu(s) = \begin{cases} \mu'(s') & : \text{ if } s = s' \setminus L \\ 0 & : \text{ otherwise} \end{cases}$
6. $x \xrightarrow{\alpha}_\sigma \mu$ if $\sigma(x) \xrightarrow{\alpha}_\sigma \mu$

Fig. 4. Operational semantics for *PCCS*

are different from ours as they distinguish between probabilistic and non-deterministic processes, whereas we do not.

5.1. *Operational semantics for Probabilistic CCS*

We give an operational semantics for *PCCS* based on pointed probabilistic transition systems. Let σ be a declaration. We define a probabilistic transition system $(Stmt, \rightarrow_\sigma)$ where

$$\rightarrow_\sigma \;\subseteq\; Stmt \times Act \times \mathcal{D}_1(Stmt)$$

is the smallest relation satisfying the rules of Figure 4. Here, we write $s \xrightarrow{\alpha}_\sigma \mu$ instead of $(s, \alpha, \mu) \in \rightarrow_\sigma$. The operational semantics assigns to each program $\mathcal{P} =< \sigma, s >$ the pointed transition system $(Stmt, \rightarrow_\sigma, s)$. In what follows, we identify each program $\mathcal{P}$ with its operational meaning.

Example 5.1. The operational semantics of the programs $< \sigma, s >$ and $< \sigma, s' >$ where

$$s \stackrel{\text{def}}{=} \alpha_{\frac{1}{3}}.\beta_1.nil + \alpha_{\frac{2}{3}}.nil, \quad s' \stackrel{\text{def}}{=} \alpha_{\frac{1}{2}}.\beta_1.nil + \alpha_{\frac{1}{2}}.nil$$

(and σ is an arbitrary declaration) are the pointed transition systems shown in Figure 2. Figure 5 shows the pointed transition systems of the program $< \sigma, (s_1 \| s_2) \setminus L >$ where

$$s_1 \stackrel{\text{def}}{=} \alpha_{\frac{1}{4}}.\beta_1.nil \;+\; \alpha_{\frac{3}{4}}.\overline{\alpha}_1.nil, \quad s_2 \stackrel{\text{def}}{=} (\overline{\alpha}_{\frac{1}{3}}.\overline{\beta}_1.nil \;+\; \overline{\alpha}_{\frac{2}{3}}.\alpha_1.nil) \;\oplus\; \gamma_1.nil$$

and $L \stackrel{\text{def}}{=} \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}$. In the picture, $t_1 \otimes t_2$ stands for $(t_1 \| t_2) \setminus L$. The γ-transition of $s_1 \otimes s_2$ represents the case where the γ-transition of s_2 is chosen non-deterministically. Thus, $s_1 \otimes s_2$ can make a γ-move where s_1 does not participate, i.e. does not change its local state. The τ-transition of $s_1 \otimes s_2$ stands for the synchronization of α and $\overline{\alpha}$. For instance, with probability $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$, s_1 moves to the local state $\beta_1.nil$ and s_2 to $\overline{\beta}_1.nil$. In the global states $s_1 \otimes nil$, $\beta_1.nil \otimes \alpha_1.nil$ and $\overline{\alpha}_1.nil \otimes \overline{\beta}_1.nil$ no actions are possible because of the restriction operator while in the global states $\beta_1.nil \otimes \overline{\beta}_1.nil$ and $\overline{\alpha}_1.nil \otimes \alpha_1.nil$ further synchronizations take place. Figure 6 shows the pointed transition system of the

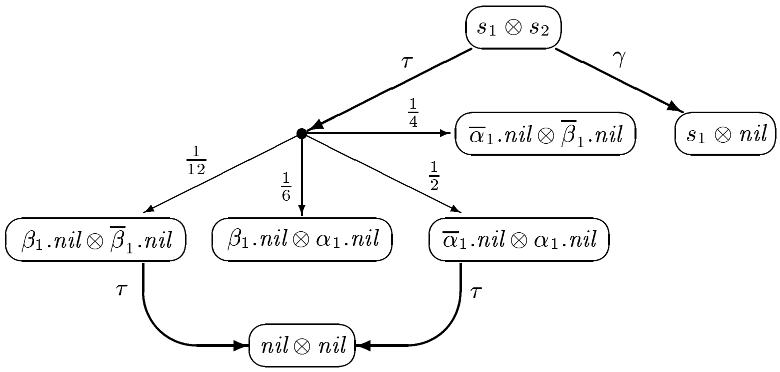


Fig. 5. The operational semantics of $< \sigma, (s_1 \| s_2) \setminus \{\alpha, \beta\} >$

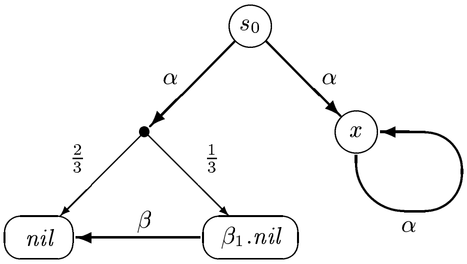


Fig. 6. The operational semantics of $< \sigma_0, (\alpha_{\frac{1}{3}}.\beta_1.nil + \alpha_{\frac{2}{3}}.nil) \oplus x >$ where $\sigma_0(x) = \alpha_1.x$

recursive program $\mathcal{P} =< s_0, \sigma_0 >$ where $s_0 \stackrel{\text{def}}{=} s \oplus x$ and $\sigma_0(x) \stackrel{\text{def}}{=} \alpha_1.x$ (and s is as above). ■

It can easily be shown that all operators preserve bisimilarity and the simulation preorder. More precisely, if σ is a declaration then we define $\sim^\sigma$ and $\sqsubseteq^\sigma_{\text{sim}}$ as binary relation on *Stmt* by $s \sim^\sigma s'$ iff $< \sigma, s > \sim < \sigma, s' >$ and $s \sqsubseteq_{\rm sim}^\sigma s'$ iff $< \sigma, s > \sqsubseteq_{\rm sim} < \sigma, s' >$. Let $\theta = \sim^\sigma$ or $\theta = \sqsubseteq_{\rm sim}^\sigma$. Then:

1. If $s_i \ \theta \ s_i'$, $i \in I$, then $\sum \alpha_{p_i}.s_i \ \ \theta \ \ \sum_{p_i}.s_i'$.
2. If $s_1 \ \theta \ s_1'$ and $s_2 \ \theta \ s_2'$ then $s_1 \oplus s_2 \ \ \theta \ \ s_1' \oplus s_2'$.
3. If $s_1 \ \theta \ s_1'$ and $s_2 \ \theta \ s_2'$ then $s_1 \| s_2 \ \ \theta \ \ s_1' \| s_2'$.
4. If $s \ \theta \ s'$ then $s[\lambda] \ \ \theta \ \ s'[\lambda]$.
5. If $s \ \theta \ s'$ then $s \setminus L \ \ \theta \ \ s' \setminus L$.
6. $x \ \theta \ \sigma(x)$

5.2. *Denotational semantics on* $\mathbf{M}$ *and* $\mathbf{D}$

Recall that $\mathbf{M}$ denotes the solution of the ultrametric equation in *CUM* (Notation 4.20), and $\mathbf{D}$ the solution of the equation in $CONT_\perp$ (Notation 4.11). We give a denotational semantics for *GPCCS* on $\mathbf{M}$, which is fully abstract w.r.t. bisimulation equivalence, and a denotational semantics for *PCCS* on $\mathbf{D}$, which is fully abstract w.r.t. the simulation preorder. For this we need non-expansive/contracting semantic operators on $\mathbf{M}$ and continuous semantic operators on $\mathbf{D}$. In the sequel, $X = \mathbf{M}$ or $X = \mathbf{D}$.

The process ***nil***: The process *nil* is modelled by $\emptyset$ in $\mathbf{M}$ and by $\perp_{\mathbf{D}} = \{\perp\}$ in $\mathbf{D}$.

Nondeterministic choice: Nondeterministic choice on $\mathbf{M}$ and $\mathbf{D}$ is modelled by set-theoretic union. The union is continuous as an operator on $\mathbf{D}$, and non-expansive when considered as an operator on $\mathbf{M}$.

(Guarded) probabilistic choice: Let $\alpha \in Act$ and $(p_i)_{i \in I}$ be a countable family of real numbers $p_i > 0$ with $\sum_{i \in I} p_i = 1$. Let $(x_i)_{i \in I}$ be a family in X. We put

$$\sum_{i \in I} \alpha_{p_i}.x_i \stackrel{\rm def}{=} \begin{cases} \{(\alpha, E_\mu)\} & : \ \text{if } X = \mathbf{M} \\ \{(\alpha, E_\mu)\}^* & : \ \text{if } X = \mathbf{D} \end{cases} \quad \text{where } \mu \in \mathcal{D}_1(X) \text{ is given by} \mu(x) \stackrel{\rm def}{=} \sum_{\substack{i \in I \\ x_i = x}} p_i.$$

The operator $\sum$ is contracting on $\mathbf{M}$ and continuous on $\mathbf{D}$. Let $(x_i)_{i \in I}$, $(x_i')_{i \in I}$ be families in $\mathbf{M}$. Then,

$$d\left(\sum_{i \in I} \alpha_{p_i}.x_i, \sum_{i \in I} \alpha_{p_i}.x_i' \right) \ \ \leq \ \ \frac{1}{2} \cdot \max\{d(x_i, x_i') : i \in I\}$$

Let $(x_i)_{i \in I}$ be a family in $\mathbf{D}$ and $x_i = \bigsqcup X_i$ where X_i is a directed subset of $\mathbf{D}$. Then,

$$\sum_{i \in I} \alpha_{p_i}.x_i \ \ = \ \ \bigsqcup \left\{ \sum_{i \in I} \alpha_{p_i}.y_i \ : \ y_i \in X_i, i \in I \right\}.$$

We define semantic operators for modelling restriction, relabelling and parallelism as fixed points of suitable operators. This reflects the recursive nature of restriction, relabelling and parallelism, cf. Milner's expansion law (Milner 1989) for parallelism.

Relabelling: If λ is a relabelling function then $F_\lambda^X : (X \to X) \to (X \to X)$ is given by

$$\begin{aligned} F_\lambda^{\mathbf{M}}(f)(x) &\stackrel{\rm def}{=} \{ (\lambda(\alpha), \mathcal{E}_1(f)(E)) \ : \ (\alpha, E) \in x \}, \\ F_\lambda^{\mathbf{D}}(f)(x) &\stackrel{\rm def}{=} \{ (\lambda(\alpha), \mathcal{E}_1(f)(E)) \ : \ (\alpha, E) \in x \}^*. \end{aligned}$$

Restriction: Let L be a subset of *Act* with $L = \overline{L}$. We define $F_L^X : (X \to X) \to (X \to X)$ as follows:

$$\begin{array}{lcl} F_L^{\mathbf{M}}(f)(x) & \stackrel{\text{def}}{=} & \{ \, (\alpha, \mathcal{E}_1(f)(E)) \; : \; (\alpha, E) \in x, \; \alpha \notin L \, \}, \\ F_L^{\mathbf{D}}(f)(x) & \stackrel{\text{def}}{=} & \{ \, (\alpha, \mathcal{E}_1(f)(E)) \; : \; (\alpha, E) \in x, \; \alpha \notin L \, \}^*. \end{array}$$

Lemma 5.2. The operators $F_\lambda^{\mathbf{M}}$ and $F_L^{\mathbf{M}}$ are contracting and the unique fixed points are non-expansive. The operators $F_\lambda^{\mathbf{D}}$ and $F_L^{\mathbf{D}}$ are continuous and have unique fixed points. These are continuous.

Proof. It is easy to see that each of the four operators satisfies $F(\pi_n^X \circ f) = \pi_{n+1}^X \circ F(f)$. Hence, by Lemma 4.21, $F_\lambda^{\mathbf{M}}$ and $F_L^{\mathbf{M}}$ have unique fixed points. In order to see that these are non-expansive we observe that F maps non-expansive functions to non-expansive functions. Since the set of non-expansive functions $\mathbf{M} \to \mathbf{M}$ is a closed subspace of the complete metric space of all functions $\mathbf{M} \to \mathbf{M}$, the unique fixed points are non-expansive.

It is easy to see that $F_\lambda^{\mathbf{D}}$ and $F_L^{\mathbf{D}}$ are continuous (using the local continuity of $\mathcal{E}_1$). Then, by Lemma 4.12, $F_\lambda^{\mathbf{D}}$ and $F_L^{\mathbf{D}}$ have unique fixed points. These are continuous since both map continuous functions to continuous functions and since the set of continuous function is lub-closed. □

The unique fixed points of the operators F_λ^X and F_L^X yield non-expansive, resp. continuous, semantic operators $x \mapsto x[\lambda]$ and $x \mapsto x \setminus L$, on $\mathbf{M}$ and $\mathbf{D}$ for modelling relabelling and restriction.

Parallel composition: In the definition of the semantic parallel operators we use a result of Heckmann (Heckmann 1995) stating the existence of a unique evaluation $E_1 * E_2 \in \mathcal{E}_1(X \times X)$ with

$$E_1(U) * E_2(V) = (E_1 * E_2)(U \times V)$$

for all opens U, V of X and for all evaluations E_1, $E_2 \in \mathcal{E}_1(X)$. Here, X is an arbitrary topological space and $X \times X$ the product space (with the sets $U \times V$, U, $V \in \Omega(X)$, as its basic opens). Note that (Heckmann 1995) deals with $\mathcal{E}(\cdot)$ instead of $\mathcal{E}_1(\cdot)$, but it is clear that $(E_1 * E_2)(X \times X) = 1$ for E_1, $E_2 \in \mathcal{E}_1(X)$. Thus, $E_1 * E_2 \in \mathcal{E}_1(X \times X)$. Clearly, if μ_1, $\mu_2 \in \mathcal{D}_1(X)$ then $E_{\mu_1} * E_{\mu_2} = E_{\mu_1 * \mu_2}$ where the composition $\mu_1 * \mu_2$ for distributions is defined as in Section 2.1. It is shown in (Heckmann 1995) that for every cpo D the product $*$ is a continuous operator $\mathcal{E}_1(D) \times \mathcal{E}_1(D) \to \mathcal{E}_1(D \times D)$.

In the metric approach, where $\mathcal{E}_1(\mathbf{M})$ is a completion of $\mathcal{D}_1(\mathbf{M})$, the product of evaluations can be defined without using the result of (Heckmann 1995), as the operator $* : \mathcal{E}_1(\mathbf{M}) \times \mathcal{E}_1(\mathbf{M}) \to \mathcal{E}_1(\mathbf{M} \times \mathbf{M})$ which is defined as the canonical extension of the non-expansive operator $\mathcal{D}_1(\mathbf{M}) \times \mathcal{D}_1(\mathbf{M}) \to \mathcal{D}_1(\mathbf{M} \times \mathbf{M})$, $(\mu_1, \mu_2) \mapsto \mu_1 * \mu_2$. An alternative definition of the product (which leads to the same operator) uses Lemma A.11: if E_1, E_2 are evaluations on $\mathbf{M}$ then $E_1 * E_2$ denotes the unique evaluation on $\mathbf{M} \times \mathbf{M}$ such that for all open balls B, B' of $\mathbf{M}$:

$$(E_1 * E_2)(B \times B') \; = \; E_1(B) \cdot E_2(B').$$

Note that the open balls of $\mathbf{M} \times \mathbf{M}$ have the form $B \times B'$ where B, B' are open balls (of the same radius).

We use the following notations. If $f : X \times X \to X$ is a function and $x_0, y_0 \in X$ then we define $f(x_0, \cdot), f(\cdot, y_0) : X \to X$ by $f(x_0, \cdot)(y) \stackrel{\text{def}}{=} f(x_0, y)$ and $f(\cdot, y_0)(x) \stackrel{\text{def}}{=} f(x, y_0)$. In the case when $X = \mathbf{M}$ and f is non-expansive $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are non-expansive. In the case when $X = \mathbf{D}$ and f is continuous then $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are continuous. We define $F_{\|}^X : (X \times X \to X) \to (X \times X \to X)$ by

$$\begin{aligned} F_{\|}^{\mathbf{M}}(f)(x,y) &\stackrel{\text{def}}{=} F_1^{\mathbf{M}}(f)(x,y) \cup F_2^{\mathbf{M}}(f)(x,y) \cup F_{\text{syn}}^{\mathbf{M}}(f)(x,y), \\ F_{\|}^{\mathbf{D}}(f)(x,y) &\stackrel{\text{def}}{=} \left(F_1^{\mathbf{D}}(f)(x,y) \cup F_2^{\mathbf{D}}(f)(x,y) \cup F_{\text{syn}}^{\mathbf{D}}(f)(x,y)\right)^* \end{aligned}$$

where

$$\begin{aligned} F_1^X(f)(x,y) &\stackrel{\text{def}}{=} \{\, (\alpha, \mathcal{E}_1(f(\cdot, y))(E)) \;:\; (\alpha, E) \in x \,\}, \\ F_2^X(f)(x,y) &\stackrel{\text{def}}{=} \{\, (\alpha, \mathcal{E}_1(f(x, \cdot))(E)) \;:\; (\alpha, E) \in y \,\}, \\ F_{\text{syn}}^X(f)(x,y) &\stackrel{\text{def}}{=} \{(\tau, \mathcal{E}_1(f)(E_1 * E_2)) : (\alpha, E_1) \in x, (\overline{\alpha}, E_2) \in y \text{ for some } \alpha \neq \tau\}. \end{aligned}$$

Lemma 5.3. $F_{\|}^{\mathbf{M}}$ is contracting, $F_{\|}^{\mathbf{D}}$ is continuous. $F_{\|}^{\mathbf{M}}$ and $F_{\|}^{\mathbf{D}}$ have unique fixed points. The unique fixed point of $F_{\|}^{\mathbf{M}}$ is non-expansive on $\mathbf{M}$, the unique fixed point of $F_{\|}^{\mathbf{D}}$ is continuous on $\mathbf{D}$.

Proof. It is easy to see that $F(\pi_n^X \circ f) = \pi_{n+1}^X \circ F(f)$ where F is one of the operators F_{syn}^X, F_1^X or F_2^X. Since $\pi_n^X(x \cup y) = \pi_n^X(x) \cup \pi_n^X(y)$ we get $F_{\|}^X(f)(\pi_n^X \circ f) = \pi_{n+1}^X \circ F_{\|}^X(f)$.

In the case $X = \mathbf{M}$ we use Lemma 4.21 and get that $F_{\|}^{\mathbf{M}}$ is contracting. In order to see that the unique fixed point of $F_{\|}^{\mathbf{M}}$ is non-expansive we observe that $F_{\|}^{\mathbf{M}}$ can be restricted to a contracting operator on non-expansive functions $\mathbf{M} \times \mathbf{M} \to \mathbf{M}$. Hence, its unique fixed point is non-expansive.

The continuity of $F_{\|}^{\mathbf{D}}$ is an easy verification (uses the local continuity of $\mathcal{E}_1(\cdot)$ and the continuity of the product). Using Lemma 4.12 we obtain the uniqueness of a fixed point of $F_{\|}^{\mathbf{D}}$. This operator is continuous since $F_{\|}^{\mathbf{D}}$ can be restricted to an operator on continuous functions $\mathbf{D} \times \mathbf{D} \to \mathbf{D}$. $\square$

The unique fixed point of $F_{\|}^X$ yields an operator $X \times X \to X$, $(x, y) \mapsto x \| y$, which is non-expansive as an operator on $X = \mathbf{M}$ and which is continuous as an operator on $\mathbf{D}$.

The following lemma builds the foundation for our main results (Theorem 5.6 and 5.8) which state the full abstraction of the denotational semantics that we obtain on $\mathbf{M}$ and $\mathbf{D}$ using the semantic operators as above.

Notation 5.4. Let $\mathit{probtree}_\sigma(s)$ denote the bisimulation equivalence class of the operational meaning of the program $< \sigma, s >$, i.e. of the pointed transition system $(\mathit{Stmt}, \to_\sigma, s)$.

Recall that $\imath_{\mathbf{D}} : \mathit{ProbTree} \to \mathbf{D}$ is the unique function such that $\imath_{\mathbf{D}}(\mathcal{T}) = \{(\alpha, E_{\mathcal{D}_1(\imath)(\mu)}) : \mathcal{T} \xrightarrow{\alpha} \mu\}^*$ (Theorem 4.13), $\imath_{\mathbf{M}} : \mathit{ProbTree} \to \mathbf{M}$ the unique function such that $\imath_{\mathbf{M}}(\mathcal{T}) = \{(\alpha, E_{\mathcal{D}_1(\imath)(\mu)}) : \mathcal{T} \xrightarrow{\alpha} \mu\}$ (Theorem 4.23).

Lemma 5.5. Let $X = \mathbf{M}$ or $X = \mathbf{D}$. Then, for each declaration σ:

0. $\imath_{\mathbf{M}}(\mathit{probtree}_\sigma(\mathit{nil})) = \emptyset$, $\imath_{\mathbf{D}}(\mathit{probtree}_\sigma(\mathit{nil})) = \bot_{\mathbf{D}}$.

1. $\imath_X\left(\sum_{i\in I}\ \alpha_{p_i}.s_i\right) \;=\; \sum_{i\in I}\ \alpha_{p_i}.\imath_X(\mathit{probtree}_\sigma(s_i))$
2. $\imath_X(\mathit{probtree}_\sigma(s_1\ \oplus\ s_2)) \;=\; \imath_X(\mathit{probtree}_\sigma(s_1)) \,\cup\, \imath_X(\mathit{probtree}_\sigma(s_2))$
3. $\imath_X(\mathit{probtree}_\sigma(\ s_1 \parallel s_2\ ) \;=\; \imath_X(\mathit{probtree}_\sigma(s_1)) \parallel \imath_X(\mathit{probtree}_\sigma(s_2))$
4. $\imath_X(\mathit{probtree}_\sigma(s[\lambda])) \;=\; \imath_X(\mathit{probtree}_\sigma(s))[\lambda]$
5. $\imath_X(\mathit{probtree}_\sigma(s\setminus L)) \;=\; \imath_X(\mathit{probtree}_\sigma(s))\setminus L$
6. $\imath_X(\mathit{probtree}_\sigma(x)) \;=\; \imath_X(\mathit{probtree}_\sigma(\sigma(x)))$

Proof. 0., 1. and 2. are clear. 6. is clear since the transition systems of $<\sigma, x>$ and $<\sigma,\sigma(x)>$ are bisimilar (and hence, by Theorem 3.3, $\mathit{probtree}_\sigma(x) = \mathit{probtree}_\sigma(\sigma(x))$).

We shortly write π_n and $\imath$ rather than π_n^X and $\imath_X$. In what follows, we use the star notation for subsets of $\mathit{Act}\times\mathbf{M}$ and for subsets of $\{\bot\}\cup\mathit{Act}\times\mathbf{D}$. When dealing with $\mathbf{D}$, A^* denotes the Scott-closure of A (if $A\neq\emptyset$) and $\emptyset^* = \bot_{\mathbf{D}}$, as before. When dealing with $\mathbf{M}$, we put $A^* \stackrel{\mathrm{def}}{=} A$. Then,

$$\imath(\mathit{probtree}_\sigma(s)) \;=\; \left\{(\alpha, E_{\mathcal{D}_1(\imath\circ \mathit{probtree}_\sigma)(\mu)}) : s\xrightarrow{\alpha}_\sigma\mu\right\}^*$$

for all $s\in\mathit{Stmt}$. We show 3. By induction on n we prove that

$$\pi_n(\imath(\mathit{probtree}_\sigma(s_1\|s_2))) \;=\; \pi_n(\imath(\mathit{probtree}_\sigma(s_1))) \parallel \pi_n(\imath(\mathit{probtree}_\sigma(s_2)))$$

for all $s_1,\ s_2\in\mathit{Stmt}$. Then, by the non-expansitivity/continuity of $\|$ and the fact that $x = \lim\pi_n(x)$ in $\mathbf{M}$ and $x = \bigsqcup\pi_n(x)$ in $\mathbf{D}$ we obtain

$$\imath(\mathit{probtree}_\sigma(s_1\|s_2)) \;=\; \imath(\mathit{probtree}_\sigma(s_1)) \parallel \imath(\mathit{probtree}_\sigma(s_2)).$$

The basis of induction ($n=0$) is clear as $\emptyset\|\emptyset=\emptyset$ in $\mathbf{M}$ and $\bot_{\mathbf{D}}\|\bot_{\mathbf{D}} = \bot_{\mathbf{D}}$ in $\mathbf{D}$. In the induction step $n\Longrightarrow n+1$ we suppose that

$$\pi_n(\imath(\mathit{probtree}_\sigma(t_1\|t_2))) \;=\; \pi_n(\imath(\mathit{probtree}_\sigma(t_1))) \parallel \pi_n(\imath(\mathit{probtree}_\sigma(t_2)))$$

for all $t_1,\ t_2\in\mathit{Stmt}$. Let $s_1,\ s_2\in\mathit{Stmt}$. Since $E_{\mathcal{D}_1(f)(\mu)} = \mathcal{E}_1(f)(E_\mu)$ (Remark 4.2) and $E_{\mu_1*\mu_2} = E_{\mu_1} * E_{\mu_2}$ we have

$$\begin{array}{rl} \imath(\mathit{probtree}_\sigma(s_1\|s_2)) \;=\; & \{(\alpha, E_{\mathcal{D}_1(\imath\circ\mathit{probtree}_\sigma)(\mu)}) \ :\ s_1\|s_2\xrightarrow{\alpha}_\sigma\mu\}^* \\ =\ (& \{(\alpha, E_{\mathcal{D}_1(\imath\circ\mathit{probtree}_\sigma(\cdot\|s_2))(\mu)}) \ :\ s_1\xrightarrow{\alpha}_\sigma\mu\} \\ \cup & \{(\alpha, E_{\mathcal{D}_1(\imath\circ\mathit{probtree}_\sigma(s_1\|\cdot))(\mu)}) \ :\ s_2\xrightarrow{\alpha}_\sigma\mu\} \\ \cup & \{(\tau, E_{\mathcal{D}_1(\imath\circ\mathit{probtree}_\sigma(\cdot\|\cdot))(\mu_1*\mu_2)}) : s_1\xrightarrow{\alpha}_\sigma\mu_1, s_2\xrightarrow{\overline{\alpha}}_\sigma\mu_2, \alpha\neq\tau\})^*. \end{array}$$

We define functions $\varphi,\ \psi : \mathit{Stmt}\times\mathit{Stmt}\to X$ by

$$\begin{array}{rcl} \varphi(t_1,t_2) & \stackrel{\mathrm{def}}{=} & \pi_n(\imath(\mathit{probtree}_\sigma(t_1))) \parallel \pi_n(\imath(\mathit{probtree}_\sigma(t_2))), \\ \psi(t_1,t_2) & \stackrel{\mathrm{def}}{=} & \pi_n(\imath(\mathit{probtree}_\sigma(t_1\ \|\ t_2))). \end{array}$$

We have

$$\pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_1\|s_2))) \;=\; \bigcup_{\alpha\in\mathit{Act}} \{(\alpha, E_\nu) : \nu\in\mathcal{M}^\alpha\}^*$$

where $\mathcal{M}^\alpha \stackrel{\text{def}}{=} \mathcal{M}_1^\alpha \cup \mathcal{M}_2^\alpha$ if $\alpha \neq \tau$ and $\mathcal{M}^\tau \stackrel{\text{def}}{=} \mathcal{M}_1^\tau \cup \mathcal{M}_2^\tau \cup \mathcal{M}_{\text{syn}}$,

$$\begin{array}{lcl} \mathcal{M}_1^\alpha & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\psi(\cdot, s_2))(\mu) : s_1 \stackrel{\alpha}{\rightarrow}_\sigma \mu\}, \\ \mathcal{M}_2^\alpha & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\psi(s_1, \cdot)(\mu) : s_2 \stackrel{\alpha}{\rightarrow}_\sigma \mu\}, \\ \mathcal{M}_{\text{syn}} & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\psi)(\mu_1 * \mu_2) : s_1 \stackrel{\alpha}{\rightarrow}_\sigma \mu_1,\ s_2 \stackrel{\overline{\alpha}}{\rightarrow}_\sigma \mu_2, \alpha \neq \tau\}. \end{array}$$

On the other hand,

$$\pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_1))) \parallel \pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_2))) \;=\; \bigcup_{\alpha \in Act} \{(\alpha, E_\nu) : \nu \in \mathcal{N}^\alpha\ \}^*$$

where $\mathcal{N}^\alpha \stackrel{\text{def}}{=} \mathcal{N}_1^\alpha \cup \mathcal{N}_2^\alpha$ if $\alpha \neq \tau$ and $\mathcal{N}^\tau \stackrel{\text{def}}{=} \mathcal{N}_1^\tau \cup \mathcal{N}_2^\alpha \cup \mathcal{N}_{\text{syn}}$,

$$\begin{array}{lcl} \mathcal{N}_1^\alpha & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\varphi(\cdot, s_2))(\mu) : s_1 \stackrel{\alpha}{\rightarrow}_\sigma \mu\}, \\ \mathcal{N}_2^\alpha & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\varphi(s_1, \cdot)(\mu) : s_2 \stackrel{\alpha}{\rightarrow}_\sigma \mu\}, \\ \mathcal{N}_{\text{syn}} & \stackrel{\text{def}}{=} & \{\mathcal{D}_1(\varphi)(\mu_1 * \mu_2) : s_1 \stackrel{\alpha}{\rightarrow}_\sigma \mu_1,\ s_2 \stackrel{\overline{\alpha}}{\rightarrow}_\sigma \mu_2, \alpha \neq \tau\}. \end{array}$$

The induction hypothesis yields $f(t_1, t_2) = g(t_1, t_2)$ for all $t_1, t_2 \in \mathit{Stmt}$. Thus, $\mathcal{M}_1^\alpha = \mathcal{N}_1^\alpha$, $\mathcal{M}_2^\alpha = \mathcal{N}_2^\alpha$ and $\mathcal{M}_{\text{syn}} = \mathcal{N}_{\text{syn}}$. Thus,

$$\pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_1 \| s_2))) \;=\; \pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_1))) \parallel \pi_{n+1}(\imath(\mathit{probtree}_\sigma(s_2))).$$

The proofs of 4. and 5. are similar to the proof of 3. □

5.2.1. *A denotational semantics on* **M** We define the denotational semantics as the unique fixed point of a contracting operator. Let σ be a guarded declaration. We define the operator $\Psi_\sigma^{\mathbf{M}} : (\mathit{Stmt} \to \mathbf{M}) \to (\mathit{Stmt} \to \mathbf{M})$ as follows:

0. $\Psi_\sigma^{\mathbf{M}}(f)(\mathit{nil}) \stackrel{\text{def}}{=} \emptyset$
1. $\Psi_\sigma^{\mathbf{M}}(f)\left(\sum_{i \in I} \alpha_{p_i}.s_i\ \right) \stackrel{\text{def}}{=} \sum_{i \in I} \alpha_{p_i}.\Psi_\sigma^{\mathbf{M}}(f)(s_i)$
2. $\Psi_\sigma^{\mathbf{M}}(f)(s_1 \oplus s_2) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{M}}(f)(s_1) \ \cup\ \Psi_\sigma^{\mathbf{M}}(f)(s_2)$
3. $\Psi_\sigma^{\mathbf{M}}(f)(s_1 \parallel s_2) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{M}}(f)(s_1) \parallel \Psi_\sigma(f)(s_2)$
4. $\Psi_\sigma^{\mathbf{M}}(f)(s[\lambda]) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{M}}(f)(s)[\lambda]$
5. $\Psi_\sigma^{\mathbf{M}}(f)(s \setminus L) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{M}}(f)(s) \setminus L$
6. $\Psi_\sigma^{\mathbf{M}}(f)(x) \stackrel{\text{def}}{=} f(\sigma(x))$

By the results of (Baier and Majster-Cederbaum 1994), $\Psi_\sigma^{\mathbf{M}}$ is contracting and hence has a unique fixed point $f_\sigma^{\mathbf{M}}$. We obtain a denotational semantics for *GPCCS* on **M**:

$$\mathit{Me}^{\mathbf{M}} \ :\ \mathit{GPCCS} \to \mathbf{M}, \quad \mathit{Me}^{\mathbf{M}}(<\sigma, s>) \stackrel{\text{def}}{=} f_\sigma^{\mathbf{M}}(s)$$

Theorem 5.6. The denotational semantics $\mathit{Me}^{\mathbf{M}} : \mathit{GPCCS} \to \mathbf{M}$ is fully abstract w.r.t. the operational semantics and bisimulation. For all $\mathcal{P}$, $\mathcal{P}' \in \mathit{GPCCS}$:

(a) $\mathit{Me}^{\mathbf{M}}(\mathcal{P}) \;=\; \mathit{probtree}(\mathcal{P})$
(b) $\mathcal{P} \sim \mathcal{P}'$ iff $\mathit{Me}^{\mathbf{M}}(\mathcal{P}) \;=\; \mathit{Me}^{\mathbf{M}}(\mathcal{P}')$.

Here, *ProbTree* is considered as a subspace of **M** (Theorem 4.23).

Proof. Using Lemma 5.5 it can be shown by structural induction on the syntax of $s \in \mathit{Stmt}$ that $\Psi_\sigma^{\mathbf{M}}(\mathit{probtree}_\sigma)(s) = \mathit{probtree}_\sigma(s)$. By the uniqueness of $f_\sigma^{\mathbf{M}}$ as a fixed point of $\Psi_\sigma^{\mathbf{M}}$ we get $f_\sigma^{\mathbf{M}} = \mathit{probtree}_\sigma$. Hence, $Me^{\mathbf{M}}(\mathcal{P}) = \mathit{probtree}(\mathcal{P})$. Then, Theorem 3.3 can be applied which yields (b). □

Example 5.7. The denotational semantics of the program $< \sigma_0, s_0 >$ of Example 5.1 in $\mathbf{M}$ is $x \cup y$ where y is $\{(\alpha, E_\mu)\}$ and μ the unique distribution with $\mu(\emptyset) = 2/3$, $\mu(y_\beta) = 1/3$, $y_\beta = \{(\beta, E_{\mu^1_\emptyset})\}$ and x is the unique element in $\mathbf{M}$ such that $x = \{(\alpha, E_{\mu^1_x})\}$.[‖] ■

5.2.2. *A denotational semantics on* **D** We define the denotational semantics as the least fixed point of a continuous operator. Let σ be a declaration. We define the operator $\Psi_\sigma^{\mathbf{D}} : (\mathit{Stmt} \to \mathbf{D}) \to (\mathit{Stmt} \to \mathbf{D})$ as follows:

0. $\Psi_\sigma^{\mathbf{D}}(f)(\mathit{nil}) \stackrel{\text{def}}{=} \bot_{\mathbf{D}}$
1. $\Psi_\sigma^{\mathbf{D}}(f)\left(\sum_{i\in I} \alpha_{p_i}.s_i\right) \stackrel{\text{def}}{=} \sum_{i\in I} \alpha_{p_i}.\Psi_\sigma^{\mathbf{D}}(f)(s_i)$
2. $\Psi_\sigma^{\mathbf{D}}(f)(s_1 \oplus s_2) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{D}}(f)(s_1) \cup \Psi_\sigma^{\mathbf{D}}(f)(s_2)$
3. $\Psi_\sigma^{\mathbf{D}}(f)(s_1 \parallel s_2) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{D}}(f)(s_1) \parallel \Psi_\sigma^{\mathbf{D}}(f)(s_2)$
4. $\Psi_\sigma^{\mathbf{D}}(f)(s[\lambda]) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{D}}(f)(s)[\lambda]$
5. $\Psi_\sigma^{\mathbf{D}}(f)(s \setminus L) \stackrel{\text{def}}{=} \Psi_\sigma^{\mathbf{D}}(f)(s) \setminus L$
6. $\Psi_\sigma^{\mathbf{D}}(f)(x) \stackrel{\text{def}}{=} f(\sigma(x))$

By the results of (Baier and Majster-Cederbaum 1994), $\Psi_\sigma^{\mathbf{D}}$ is continuous, and hence has a least fixed point $f_\sigma^{\mathbf{D}}$. We get a denotational semantics for *PCCS* on **D**:

$$Me^{\mathbf{D}} \;:\; \mathit{PCCS} \to \mathbf{D}, \quad Me^{\mathbf{D}}(< \sigma, s >) \stackrel{\text{def}}{=} f_\sigma^{\mathbf{D}}(s).$$

Theorem 5.8. The denotational semantics $Me^{\mathbf{D}} : \mathit{PCCS} \to \mathbf{D}$ is fully abstract w.r.t. the operational semantics and simulation. For all $\mathcal{P}$, $\mathcal{P}' \in \mathit{PCCS}$:

(a) $Me^{\mathbf{D}}(\mathcal{P}) = \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}))$
(b) $\mathcal{P} \sqsubseteq_{\text{sim}} \mathcal{P}'$ iff $Me^{\mathbf{D}}(\mathcal{P}) \subseteq Me^{\mathbf{D}}(\mathcal{P}')$.

Here, $\imath_{\mathbf{D}}$ is as in Theorem 4.13.

Proof. Using Lemma 5.5 it can be shown that $\imath \circ \mathit{probtree}_\sigma$ is a fixed point of $\Psi_\sigma^{\mathbf{D}}$. We show the uniqueness of $f_\sigma^{\mathbf{D}}$ as a fixed point of $\Psi_\sigma^{\mathbf{D}}$. It is easy to see that $\pi_{n+1}^{\mathbf{D}} \circ \Psi_\sigma^{\mathbf{D}}(\pi_n^{\mathbf{D}} \circ f) = \pi_{n+1}^{\mathbf{D}} \circ \Psi_\sigma^{\mathbf{D}}(f)$. Hence, if f, f' are fixed points of $\Psi_\sigma^{\mathbf{D}}$ then (by induction on n) $\pi_n^{\mathbf{D}} \circ f = \pi_n^{\mathbf{D}} \circ f'$. Hence,

$$f(s) \;=\; \bigsqcup_{n\geq 0} \pi_n^{\mathbf{D}}(f(s)) \;=\; \bigsqcup_{n\geq 0} \pi_n^{\mathbf{D}}(f'(s)) \;=\; f'(s)$$

for all $s \in \mathit{Stmt}$. Therefore $f = f'$. We conclude $f_\sigma^{\mathbf{D}} = \imath_{\mathbf{D}} \circ \mathit{probtree}_\sigma$. Thus, $Me^{\mathbf{D}}(\mathcal{P}) = \imath_{\mathbf{D}}(\mathit{probtree}(\mathcal{P}))$. By Theorem 3.3 and Theorem 4.13, $\mathcal{P} \sqsubseteq_{\text{sim}} \mathcal{P}'$ iff $Me^{\mathbf{D}}(\mathcal{P}) \subseteq Me^{\mathbf{D}}(\mathcal{P}')$. □

[‖] Formally, $x = \lim x_n$ where $x_0 \stackrel{\text{def}}{=} \emptyset$, $x_{n+1} \stackrel{\text{def}}{=} \{(\alpha, E_{\mu^1_{x_n}})\}$.

Example 5.9. The denotational semantics of the program $< \sigma_0, s_0 >$ of Example 5.1 in $\mathbf{D}$ is $x \cup y$ where y is $\{(\alpha, E_\mu)\}^*$ and μ the unique distribution with $\mu(\perp_\mathbf{D}) = 2/3$, $\mu(y_\beta) = 1/3$, $y_\beta = \{(\beta, E_{\mu^1_{\perp_\mathbf{D}}})\}^*$ and x is the unique element in $\mathbf{D}$ such that $x = \{(\alpha, E_{\mu^1_x})\}^*$.[††] ■

Clearly, $\mathbf{D}$ is more "abstract" than $\mathbf{M}$ since simulation equivalence is coarser than bisimulation equivalence. Formally, we define a function $\Phi : \mathbf{M} \to \mathbf{D}$ as follows. We consider the function

$$F : (\mathbf{M} \to \mathbf{D}) \to (\mathbf{M} \to \mathbf{D}), \quad F(f)(x) \stackrel{\text{def}}{=} \{(\alpha, \mathcal{E}_1(f)(x)) : (\alpha, E) \in x\}^*.$$

It is easy to see that F is continuous and $F(\pi_n^\mathbf{D} \circ f) = \pi_{n-1}^\mathbf{D} \circ F(f)$. Thus, F satisfies the conditions of Lemma 4.12. Let $\Phi : \mathbf{M} \to \mathbf{D}$ be the unique fixed point of F. It is easy to see that Φ is a "homomorphism" w.r.t. the semantic operators on $\mathbf{M}$ and $\mathbf{D}$. Using the results of (Baier and Majster-Cederbaum 1997) it can be shown that, for fixed guarded declaration σ, $\Phi \circ f_\sigma^\mathbf{M}$ is a fixed point of $\Psi_\sigma^\mathbf{D}$. As we saw in the proof of Theorem 5.8, $f_\sigma^\mathbf{D}$ is the unique fixed point of $\Psi_\sigma^\mathbf{D}$. Hence, $\Phi \circ f_\sigma^\mathbf{M} = f_\sigma^\mathbf{D}$ which yields the "consistency result"

$$\Phi\left(Me^\mathbf{M}(\mathcal{P})\right) = Me^\mathbf{D}(\mathcal{P}).$$

for all $\mathcal{P} \in GPCCS$. For the notion of "consistency" the reader is referred to (Baier and Majster-Cederbaum 1997).

6. Conclusion

We have studied two general semantic frameworks, one rooted in domains and the other in complete ultrametric spaces, of probabilistic processes, and have shown that they are fully abstract with respect to natural probabilistic extensions of the simulation preorder and bisimulation equivalence. Furthermore, we obtained final semantics in the sense of (Rutten and Turi 1993). Our frameworks allow to combine the qualitative nondeterministic choice together with the quantitative probabilistic choice in an independent fashion. The simpler, reactive, models can be obtained as special cases by replacing the non-deterministic choice with its deterministic counter-part. Moreover, it can be seen that the usual denotational semantics of the non-probabilistic *CCS* arises as a special case of the probabilistic version by setting the distributions to the simple distributions (point masses).

To construct the denotational models we have generalised to the probabilistic setting the established category-theoretic techniques for solving domain equations for non-probabilistic processes (notably, synchronization trees). These generalised domain equations involved appropriately adjusted probabilistic powerdomain constructions. Solving the domain equation in the category of continuous domains yields a "smooth" construction, in the sense that, for example, the probabilistic powerdomain $\mathcal{E}_1(S)$ of a two-point space S is the real interval $[0, 1]$. Thus, limits can be approximated by approaching them

[††] Formally, $x = \bigsqcup x_n$ where $x_0 \stackrel{\text{def}}{=} \perp_\mathbf{D}$, $x_{n+1} \stackrel{\text{def}}{=} \{(\alpha, E_{\mu^1_{x_n}})\}^*$.

arbitrarily close. The probabilistic powerdomain of (Jones and Plotkin 1989) is also "smooth" in this sense. On the other hand, in the ultrametric case we obtain a "discrete" construction, in the sense that the two-point space lifted to the probabilistic case gives the real interval $[0, 1]$ with the discrete topology. In particular, it is not possible to get arbitrarily close to a limit. We should emphasise though that the methodology we used to derive an ultrametric model is consistent with the established methodology (in particular, the metric satisfies the intuitive property $d(x, y) \leq \frac{1}{2^n}$ iff x and y agree up to the nth step, and we obtain full abstraction for bisimulation), and that an attempt to obtain a "smooth" metric construction might mean having to go beyond the known techniques, see (Kwiatkowska and Norman 1996; Norman 1997; Kwiatkowska and Norman 1998).

We finish with a discussion of how our framework may be used to derive fully abstract denotational semantics for other probabilistic calculi. For example, in order to give a denotational semantics for a calculus that allows for general probabilistic choice (rather than guarded probabilistic choice), e.g. probabilistic *CCS* as introduced in (Yi and Larsen 1992), we have to change the underlying model and deal with probabilistic transition systems that distinguish between action and probability labelled transitions. I.e. we have to solve recursive domain equations of the form

$$X \cong \wp_* \left(Act \times X \cup \mathcal{E}_1(X) \right).$$

When dealing with a probabilistic extension of *SCCS* where the non-deterministic choice operator is replaced by a probabilistic choice operator, e.g. the calculi of (Giacalone *et al.* 1990; Tofts 1994), the operational semantics is based on "generative" transition systems (i.e. action-labelled Markov chains). In this case we have to solve equations of the form $X \cong \mathcal{E}_1(Act \times X)$ to get denotational models that are fully abstract w.r.t. bisimulation and simulation. Dealing with an operational semantics based on "reactive" transition systems (van Glabbeek *et al.* 1990), full abstraction w.r.t. (bi-)simulation can be obtained by solving domain equations of the form $X \cong Act \to \mathcal{E}_1(X)$.

As future work we should mention finding an appropriate logic for probabilistic processes and considering a language with *divergence*. We believe that the domain-theoretic approach of (Abramsky 1991) (where the Plotkin powerdomain is used rather than the Hoare powerdomain) to derive a denotational semantics for SCCS which is fully abstract with respect to partial bisimulation can be combined with our results to get a corresponding full abstraction result for a probabilistic calculus with divergence.

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Appendix A. Appendix

We give the proofs of Theorem 4.5 (see Section A.1, Theorem A.2 and Corollary A.4), Theorem 2.15 (Section A.1, Theorem A.6). and Theorem 4.7 (see Section A.2, Theorem A.12 and A.13).

A.1. *The partial order* $\preceq_{\text{sim}}$ *on* $\mathcal{D}_1(D)$

We show that $\preceq_{\text{sim}}$ (in the sense of Notation 4.4) is a partial order on distributions of a cpo D and that the function $\mathcal{D}_1(D) \to \mathcal{E}_1(D)$, $\mu \mapsto E_\mu$ is an order-preserving embedding on the partially ordered set $\mathcal{D}_1(D)$ into the cpo $\mathcal{E}_1(D)$. As a corollary we obtain Theorem 2.15 stating that similarity and bisimilarity coincide for reactive systems.

Lemma A.1. Let D be a cpo and $\mu, \mu' \in \mathcal{D}_1(D)$ such that $E_\mu \sqsubseteq E_{\mu'}$. Then, $\mu[U] \le \mu'[U]$ for all G_δ-sets U.

Proof. Recall that G_δ-sets in a topological space are countable intersections of opens. Let $U = \bigcap_{i \ge 0} U_i$ where $U_i \in \Omega(D)$. W.l.o.g. $U_1 \supseteq U_2 \supseteq \ldots$ (otherwise we deal with $U_i' = U_1 \cap \ldots \cap U_i$). Then, $A_i = D \setminus U_i$ are closed sets with $A_1 \subseteq A_2 \subseteq \ldots$ Since $E_\mu \sqsubseteq E_{\mu'}$ we have $\mu[A_i] \ge \mu'[A_i]$ for all $i \ge 1$. Let $A = \bigcup A_i = D \setminus U$. It is sufficient to show that $\mu[A] \ge \mu'[A]$. We suppose $\mu[A] < \mu'[A]$. Then,

$$\Delta \stackrel{\text{def}}{=} \frac{1}{2}(\mu'[A] - \mu[A]) \; > \; 0.$$

There exists a finite subset X of A with $\mu'[X] > \mu'[A] - \Delta$. Since X is finite there exists $i \ge 1$ with $X \subseteq A_i$. Thus,

$$\mu[A_i] \le \mu[A] \; < \mu[A] + \Delta \; = \; \mu'[A] - \Delta \; < \mu'[X] \; \le \; \mu'[A_i].$$

Contradiction. □

Theorem A.2. (cf. Theorem 4.5) Let D be a cpo, $\mu, \mu' \in \mathcal{D}_1(D)$. Then, $\mu \preceq_{\text{sim}} \mu'$ iff $E_\mu \sqsubseteq E_{\mu'}$.

Proof. For simplicity $E = E_\mu$ and $E' = E_{\mu'}$.

$\Longrightarrow$: Let $\mu \preceq_{\text{sim}} \mu'$ and $\delta : D \times D \to [0,1]$ be a weight function for (μ, μ') w.r.t. the partial order $\sqsubseteq$ on D. For all $U \in \Omega(D)$, $\delta(x,y) = 0$ if $x \in U$ and $y \in D \setminus U$. (This is because U is upward-closed and because of condition 3 of Definition 2.1.) By condition 2 of Definition 2.1:

$$\begin{aligned} E(U) \quad &= \sum_{x \in U} \mu(x) \; = \; \sum_{x \in U} \sum_{y \in U} \delta(x,y) \; = \; \sum_{y \in U} \sum_{x \in U} \delta(x,y) \\ &\le \quad \sum_{y \in U} \sum_{x \in D} \delta(x,y) \; = \; \sum_{y \in U} \mu'(y) \; = \; E'(U). \end{aligned}$$

Hence, $E \sqsubseteq E'$.

$\Longleftarrow$: We assume $E \sqsubseteq E'$. Let $\mathbf{F}$ be the set of functions $f : D \times D \to [0,1]$ such that

(1) $f(x,y) \neq 0$ for at most countably many $(x,y) \in D$.

(2) For all x, $y \in D$,

$$\mu_f(x) \stackrel{\text{def}}{=} \sum_{y \in D} f(x,y) \ \leq \ \mu(x), \qquad \mu'_f(y) \stackrel{\text{def}}{=} \sum_{x \in D} f(x,y) \ \leq \ \mu'(y).$$

(3) If $f(x,y) \neq 0$ then $x \sqsubseteq y$.

We show that there is a function $f \in \mathbf{F}$ such that $\mu_f = \mu$ and $\mu'_f = \mu'$. (Then, this function f is a weight function for (μ, μ') w.r.t. $\sqsubseteq$. Hence, $\mu \preceq_{\text{sim}} \mu'$.)

If $f \in \mathbf{F}$ then we put $\Gamma(f) \stackrel{\text{def}}{=} \sum_{x,y \in D} f(x,y)$ and

$$X_f \stackrel{\text{def}}{=} \{x \in D : \mu(x) > \mu_f(x)\}, \quad Y_f \stackrel{\text{def}}{=} \{y \in D : \mu'(y) > \mu'_f(y)\}.$$

It is easy to see that $X_f = \emptyset$ iff $\mu_f = \mu$ iff $Y_f = \emptyset$ iff $\mu'_f = \mu'$ iff $\Gamma(f) = 1$. Thus, our aim is to show the existence of a function $f \in \mathbf{F}$ with $\Gamma(f) = 1$. (Then, we may conclude that f is a weight function for (μ, μ') and therefore $\mu \preceq_{\text{sim}} \mu'$.)

If $f \in \mathbf{F}$ then we define a *f-path* to be a finite sequence $\vec{p} = (x_0, y_0, \ldots, x_n, y_n)$ in D such that $n \geq 0$ and

(i) $f(x_{i+1}, y_i) > 0$, $i = 0, 1, \ldots, n-1$
(ii) $x_i \sqsubseteq y_i$, $i = 0, 1, \ldots, n$
(iii) $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ are pairwise distinct (but $x_i = y_j$ is possible).

We define $first(\vec{p}) \stackrel{\text{def}}{=} x_0$, $last(\vec{p}) \stackrel{\text{def}}{=} y_n$, $R_f(x) \stackrel{\text{def}}{=} \{last(\vec{p}) : \vec{p}$ is a *f-path* with $first(\vec{p}) = x\}$. (Intuitively, $R_f(x)$ is the set of all $y \in D$ that can be reached from x via a f-path.)

Claim 1: If $f \in \mathbf{F}$ and $x \in X_f$ then $R_f(x) \cap Y_f \neq \emptyset$.

Proof of Claim 1: Let $f \in \mathbf{F}$ and $x \in X_f$. Let $A \stackrel{\text{def}}{=} \{z \in D : \mu(z) > 0 \vee \mu'(z) > 0\}$, $Z \stackrel{\text{def}}{=} A \setminus R_f(x)$. We define

$$U \stackrel{\text{def}}{=} \bigcap_{z \in Z} D \setminus z\downarrow .$$

As Z is countable and $z\downarrow$ Scott-closed, U is a G_δ-set. Thus, by Lemma A.1:

(*) $\mu'[U] \ \geq \ \mu[U]$

It is easy to see that $A \cap U \ = \ A \cap R_f(x)$. (For the inclusion $A \cap R_f(x) \subseteq A \cap U$ we use the fact that $R_f(x)$ is upward-closed.) Thus, $\mu[U] = \mu[R_f(x)]$ and $\mu'[U] = \mu'[R_f(x)]$. Since $x \in R_f(x)$ (as (x,x) is a f-path) and $x \in X_f$ we have

$$(**) \quad \mu[U] \ = \sum_{z \in R_f(x)} \mu(z) \ > \sum_{z \in R_f(x)} \mu_f(z) \ = \sum_{z \in R_f(x)} \sum_{y \in D} f(z,y).$$

We suppose that $R_f(x) \cap Y_f = \emptyset$. Then, $\mu'(y) = \mu'_f(y)$ for all $y \in R_f(x)$. Hence,

$$(***) \quad \mu'[U] \ = \sum_{y \in R_f(x)} \mu'(y) \ = \sum_{y \in R_f(x)} \mu'_f(y) \ = \sum_{y \in R_f(x)} \sum_{z \in D} f(z,y).$$

If $f(z,y) > 0$ and $(x_0, y_0, \ldots, x_n, y_n)$ is a f-path with $z \notin \{x_0, y_0, \ldots, x_n, y_n\}$ $x_0 = x$ and $y_n = y$ then $(x_0, y_0, \ldots, x_n, y_n, z, z)$ is a f-path. If $z = x_i$ for some i then $\vec{p} = (x_0, y_0, \ldots, x_i, x_i)$ is a f-path with $last(\vec{p}) = z$. If $z = y_i$ for some i then $\vec{p} = (x_0, y_0, \ldots, x_i, y_i)$ is a f-path with $last(\vec{p}) = z$. Thus, if $y \in R_f(x)$ and $f(z,y) > 0$ then

$z \in R_f(x)$. Hence,

$$\{(z,y) \in D \times D : f(z,y) > 0, y \in R_f(x)\} \subseteq \{(z,y) \in D \times D : f(z,y) > 0, z \in R_f(x)\}.$$

We obtain by (***) and (**):

$$\mu'[U] = \sum_{y \in R_f(x)} \sum_{z \in D} f(z,y) \leq \sum_{z \in R_f(x)} \sum_{y \in D} f(z,y) < \mu[U].$$

This contradicts (*). Thus, $R_f(x) \cap Y_f \neq \emptyset$.

By Claim 1 we obtain: If $f \in \mathbf{F}$, $\Gamma(f) < 1$ then $X_f \neq \emptyset$. Thus (by Claim 1), there exists a f-path $\vec{p}$ with $first(p) \in X_f$ and $last(\vec{p}) \in Y_f$.

Let $\vec{p} = (x_0, y_0, \ldots, x_n, y_n)$ be a f-path with $x_0 = x \in X_f$ and $y_n = y \in Y_f$. We define

$$\Delta(f,p) \stackrel{\text{def}}{=} \min \left\{\mu(x) - \mu_f(x),\ \mu'(y) - \mu'_f(y),\ f(x_{i+1}, y_i) \ : \ i = 0, \ldots, n-1\right\}$$

and $f[\vec{p}] : D \times D \to [0,1]$ by

$$f[\vec{p}](x,y) \stackrel{\text{def}}{=} \begin{cases} f(x_i,y_i) + \Delta(f,\vec{p}) & : \text{ if } (x,y) \in \{(x_i,y_i) : i = 0,1,\ldots,n\} \\ f(x_{i+1},y_i) - \Delta(f,\vec{p}) & : \text{ if } (x,y) \in \{(x_{i+1},y_i) : i = 0,1,\ldots,n-1\} \\ f(x,y) & : \text{ otherwise.} \end{cases}$$

Then, $\Delta(f,\vec{p}) > 0$, $f[\vec{p}] \in \mathbf{F}$ and

(I) $\Gamma(f[\vec{p}]) = \Gamma(f) + \Delta(f,\vec{p})$.
(II) For each $x \in D$ there is at most one $y \in D$ with $f(x,y) \neq f[\vec{p}](x,y)$.
(III) For each $y \in D$ there is at most one $x \in D$ with $f(x,y) \neq f[\vec{p}](x,y)$.
(IV) $|f(x,y) - f[\vec{p}](x,y)| \leq \Delta(f,\vec{p})$.

(II) and (III) follow by condition (iii) of f-paths. By induction on i we define a sequence $(f_i)_{i\geq 0}$ of functions $f_i \in \mathbf{F}$ as follows.

— Let f_0 be given by $f_0(x,y) \stackrel{\text{def}}{=} 0$ for all x, $y \in D$.
— Now we suppose $i \geq 1$ and that $f_0, \ldots, f_{i-1}$ are defined. If $\Gamma(f_{i-1}) = 1$ then we set $f_i \stackrel{\text{def}}{=} f_{i-1}$. Otherwise, we define

$$\Delta_i \stackrel{\text{def}}{=} \sup \left\{ \Delta(f_{i-1},\vec{p}) : \vec{p} \text{ is a } f\text{-path with } first(\vec{p}) \in X_{f_{i-1}} \text{ and } last(\vec{p}) \in Y_{f_{i-1}} \right\}.$$

We choose some f_{i-1}-path $\vec{p}_i$ with

$$first(\vec{p}_i) \in X_{f_{i-1}},\ last(\vec{p}_i) \in Y_{f_{i-1}},\ \Delta(f_{i-1},\vec{p}_i) > \Delta_i - 1/2^i.$$

We define $f_i \stackrel{\text{def}}{=} f_{i-1}[\vec{p}_i]$.

Claim 2: $\lim f_i$ exists and $\lim f_i \in \mathbf{F}$, $\Gamma(\lim f_i) = 1$.

Proof of Claim 2: By (I) we obtain $1 \geq \Gamma(f_i) = \sum_{1\leq j\leq i} \Delta(f_{j-1},\vec{p}_j)$. Thus, $\sum_j \Delta(f_{j-1},\vec{p}_j)$ is convergent. Let $\varepsilon > 0$ and $i_\varepsilon \geq 1$ such that

$$\sum_{j=i}^{k} \Delta(f_{j-1},\vec{p}_j) < \varepsilon \quad \text{for all } k \geq i \geq i_\varepsilon.$$

By (IV), for all x, $y \in D$ and $k \geq i \geq i_\varepsilon$:

$$|f_k(x,y) - f_i(x,y)| \ \leq \ \sum_{j=i+1}^{k} |f_j(x,y) - f_{j-1}(x,y)| \ \leq \ \sum_{j=i+1}^{k} \Delta(f_{j-1}, \vec{p}_j) \ < \ \varepsilon.$$

Hence, $(f_i(x,y))_{i\geq 0}$ is a Cauchy sequence. Let $f(x,y) \ = \ \lim f_i(x,y)$. Then:

(V) $\quad |f(x,y) - f_i(x,y)| \ = \ \lim_{k\to\infty} |f_k(x,y) - f_i(x,y)| \ \leq \ \varepsilon$for all $x, y \in D$, $\varepsilon > 0$, $i \geq i_\varepsilon$.

We obtain by (II), (III) and (IV):

$$\begin{array}{llcll} \text{(VI)} & |\mu_f(x) - \mu_{f_i}(x)| & \leq & \sum_{j\geq i} \Delta(f_{j-1}, \vec{p}_j) & \leq \varepsilon \quad \text{for all } \varepsilon > 0,\ i \geq i_\varepsilon,\ x \in D. \\ \text{(VII)} & |\mu'_f(y) - \mu'_{f_i}(y)| & \leq & \sum_{j\geq i} \Delta(f_{j-1}, \vec{p}_j) & \leq \varepsilon \quad \text{for all } \varepsilon > 0,\ i \geq i_\varepsilon,\ y \in D. \end{array}$$

Thus, $\mu_f(x) = \lim \mu_{f_i}(x) \leq \mu(x)$ and $\mu'_f(y) = \lim \mu'_{f_i}(y) \leq \mu'(y)$. Hence, $f \in \mathbf{F}$. We show that $\Gamma(f) = 1$. We assume that $\Gamma(f) < 1$. By Claim 1, there is some f-path $\vec{p} = (x_0, y_0, \ldots, y_n, x_n)$ with $x_0 \in X_f$ and $y_n \in Y_f$. Let $\varepsilon < 1/2 \cdot \Delta(f, \vec{p})$. By (V) (and using the fact that $f(x_{j+1}, y_j) \geq \Delta(f, \vec{p})$),

$$f_i(x_{j+1}, y_j) \ \geq \ f(x_{j+1}, y_j) - \varepsilon \ \geq \ \frac{1}{2} \cdot \Delta(f, \vec{p}) \ > \ 0, \quad j = 0, \ldots, n-1$$

for all $i \geq i_\varepsilon$. By (VI) and (VII) (and using the fact that $\mu(x_0) - \mu_f(x_0)$, $\mu'(y_n) - \mu'_f(y_n) \geq \Delta(f, \vec{p})$) we obtain:

$$\begin{array}{lclcl} \mu(x_0) - \mu_{f_i}(x_0) & \geq & \mu(x_0) - \mu_f(x_0) - \varepsilon & \geq & \frac{1}{2} \cdot \Delta(f, \vec{p}) \\ \mu'(y_n) - \mu'_{f_i}(y_n) & \geq & \mu'(y_n) - \mu'_f(y_n) - \varepsilon & \geq & \frac{1}{2} \cdot \Delta(f, \vec{p}) \end{array}$$

Thus, $\vec{p}$ is a f_i-path for all $i \geq i_\varepsilon$ and $\Delta(f_i, \vec{p}) \ \geq \ 1/2 \cdot \Delta(f, \vec{p})$. Let $j \geq 1$ such that $\Delta(f, \vec{p}) > 1/2^{j-1}$. Then, for all $i \geq i_\varepsilon$,

$$\Delta(f_i, \vec{p}) \ \geq \ \frac{1}{2} \cdot \Delta(f, \vec{p}) \ > \ \frac{1}{2^j}.$$

Hence, $\Delta_i \ \geq \ \Delta(f_{i-1}, \vec{p}) > 1/2^j$ for all $i \geq i_\varepsilon$. By definition of $\vec{p}_i$:

$$\Delta(f_{i-1}, \vec{p}_i) \ > \ \Delta_i - \frac{1}{2^i} \ > \ \frac{1}{2^j} - \frac{1}{2^i} \ \geq \ \frac{1}{2^{j+1}}$$

for all $i \geq \max\{i_\varepsilon, j+1\}$. Contradiction (as $\sum_i \Delta(f_{i-1}, \vec{p}_i)$ is convergent). $\square$

Lemma A.3. Let D be a cpo and μ, $\mu' \in \mathcal{D}_1(D)$ such that $E_\mu = E_{\mu'}$. Then, $\mu = \mu'$.

Proof. Suppose $\mu(x) \neq \mu'(x)$ for some $x \in D$. W.l.o.g. $\mu(x) < \mu'(x)$. Then,

$$\Delta \stackrel{\text{def}}{=} \mu'(x) - \mu(x) > 0.$$

Let $A = x\downarrow$. Then, A is Scott-closed. Since $\sum_{y\in A} \mu(y)$ and $\sum_{y\in A} \mu'(y)$ are convergent there exists a finite subset C_0 of A with $x \in C_0$ and

$$K \ \stackrel{\text{def}}{=} \sum_{y\in A\setminus C_0} \mu(y) \ < \ \Delta, \quad K' \ \stackrel{\text{def}}{=} \sum_{y\in A\setminus C_0} \mu'(y) \ < \ \Delta.$$

Let $C \stackrel{\text{def}}{=} C_0 \setminus \{x\}$ and $B \ \stackrel{\text{def}}{=} \ \bigcup \{ \ z\downarrow : \ z \in C \ \}$. A and B are Scott-closed (as C is finite). Since $E_\mu = E_{\mu'}$ we obtain $\mu[A] = \mu'[A]$ and $\mu[B] \ = \ \mu'[B]$. Since $C \subseteq B$ we have

$\mu[C] = \mu[B] - \mu[B \setminus C]$ and $\mu'[C] = \mu'[B] - \mu'[B \setminus C]$. Since $A = (A \setminus C_0) \cup \{x\} \cup C$ and $\mu[B \setminus C] \geq 0$ we have:

$$\mu[A] = K + \mu(x) + \mu[C] = K + \mu(x) + \mu[B] - \mu[B \setminus C]$$

$$\leq K + \mu[B] + \mu(x) < \Delta + \mu[B] + \mu(x) = \mu[B] + \mu'(x).$$

Since $B \setminus C \subseteq A \setminus C_0$ we get $\mu'[B \setminus C] \leq \mu'[A \setminus C_0] = K'$. Hence,

$$\mu'[A] = K' + \mu'(x) + \mu'[C] = K' + \mu'(x) + \mu'[B] - \mu'[B \setminus C] \geq \mu'(x) + \mu'[B].$$

Since $\mu[B] = \mu'[B]$ we obtain $\mu[A] < \mu[B] + \mu'(x) = \mu'[B] + \mu'(x) \leq \mu'[A]$. Contradiction (as $\mu[A] = \mu'[A]$). $\square$

Corollary A.4. For every partially ordered set D, $\preceq_{\text{sim}}$ is a partial order on $\mathcal{D}_1(D)$.

Proof. By Remark 2.2, $\preceq_{\text{sim}}$ is a preorder on $\mathcal{D}_1(D)$. We show the antisymmetry of $\preceq_{\text{sim}}$. Let $\overline{D}$ be the ideal completion of D. We consider D as a subspace of $\overline{D}$. For μ to be a distribution on D, we define $\overline{\mu} \in \mathcal{D}_1(\overline{D})$ as follows.

$$\overline{\mu}(x) \stackrel{\text{def}}{=} \begin{cases} \mu(x) & : \text{ if } x \in D \\ 0 & : \text{ otherwise.} \end{cases}$$

Clearly, if μ, $\mu' \in \mathcal{D}_1(D)$ and $\mu \preceq_{\text{sim}} \mu'$ then $\overline{\mu} \preceq_{\text{sim}} \overline{\mu}'$.

Let μ, μ' be distributions on D with $\mu \preceq_{\text{sim}} \mu'$ and $\mu' \preceq_{\text{sim}} \mu$. Then, $\overline{\mu} \preceq_{\text{sim}} \overline{\mu}'$ and $\overline{\mu}' \preceq_{\text{sim}} \overline{\mu}$. As $\overline{D}$ is a cpo we obtain by Theorem A.2: $E_{\overline{\mu}} \sqsubseteq E_{\overline{\mu}'}$ and $E_{\overline{\mu}'} \sqsubseteq E_{\overline{\mu}}$. Thus, $E_{\overline{\mu}} = E_{\overline{\mu}'}$. By Lemma A.3: $\overline{\mu} = \overline{\mu}'$. Hence, $\mu = \mu'$. $\square$

Lemma A.5. Let R be a preorder on a set X and μ, $\mu' \in \mathcal{D}_1(X)$. If $\mu \preceq_{\text{R}} \mu'$ and $\mu' \preceq_{\text{R}} \mu$ then $\mu[A] = \mu'[A]$ for all equivalence classes A w.r.t. the kernel $\equiv\ = R \cap R^{-1}$ of R.

Proof. Let $D = X/\equiv$ be the quotient space endowed with the partial order $\langle x \rangle \sqsubseteq \langle x' \rangle$ iff $(x, x') \in R$. (Here, $\langle y \rangle$ denotes the equivalence class of y w.r.t. $\equiv$.) Let $f : X \to D$ be the canonical projection, i.e. $f(x) = \langle x \rangle$ for all $x \in X$. For μ to be a distribution on X, we define $\mu_D : D \to [0,1]$ by

$$\mu_D(\langle x \rangle) \stackrel{\text{def}}{=} \sum_{x' \in \langle x \rangle} \mu(x') \ (= \ \mu[\ \langle y \rangle\ ]).$$

It is easy to see that, if δ is a weight function for (μ, μ') w.r.t. $\preceq_{\text{sim}}$ (where μ, $\mu' \in \mathcal{D}_1(X)$), then $\delta_D : D \times D \to [0,1]$,

$$\delta_D(\langle x \rangle, \langle y \rangle) \stackrel{\text{def}}{=} \sum_{x' \in \langle x \rangle} \sum_{y' \in \langle y \rangle} \delta(x', y')$$

is a weight function for (μ_D, μ'_D) w.r.t. $\sqsubseteq$. Hence, if μ, $\mu' \in \mathcal{D}_1(X)$, $\mu \preceq_R \mu'$ and $\mu' \preceq_R \mu$ then $\mu_D \preceq_{\text{sim}} \mu'_D$ and $\mu'_D \preceq_{\text{sim}} \mu_D$. By Corollary A.4, $\mu_D = \mu'_D$. Thus,

$$\mu[A] = \mu_D(A) = \mu'_D(A) = \mu'[A]$$

for all $A \in D = X/R$. $\square$

Theorem A.6. (cf. Theorem 2.15) If $(S, \rightarrow)$ is a reactive transition system and s, $s' \in S$ then $s \ \sim_{\text{sim}} \ s'$ implies $s \sim s'$.

Proof. We show that $\sim_{\text{sim}}$ is a bisimulation. Clearly, $\sim_{\text{sim}}$ is an equivalence relation on S. We assume S to be equipped with the preorder $R \stackrel{\text{def}}{=} \sqsubseteq_{\text{sim}}$. Let s, $s' \in S$, $s \sim_{\text{sim}} s'$ and $s \stackrel{\alpha}{\rightarrow} \mu$. There exist transitions $s' \stackrel{\alpha}{\rightarrow} \mu'$ and $s \stackrel{\alpha}{\rightarrow} \mu''$ with $\mu \preceq_R \mu'$ and $\mu' \preceq_R \mu''$. As $(S, \rightarrow)$ is reactive we have $\mu = \mu''$. By Lemma A.5, $\mu[A] = \mu'[A]$ for all $A \in S/\sim_{\text{sim}}$. □

A.2. *Probabilistic powerdomains on ultrametric spaces*

This section presents the proof that any probabilistic powerdomain $\mathcal{E}_1(M)$ of a complete ultrametric space M is complete.

Lemma A.7. Let M be an ultrametric space. Every nonempty open subset U of M can be written as disjoint union of open balls. If U is a ρ-set then U can be written as disjoint union of open balls with radius ρ.

Proof. Let U be a nonempty open subset of M. If $x \in U$ then we put

$$\rho(x) \ \stackrel{\text{def}}{=} \ \sup \{r > 0 : B(x, r) \subseteq U\}.$$

Let $\equiv$ be the following equivalence relation on U: $x \ \equiv \ y$ iff $\rho(x) = \rho(y)$. Let V be a subset of U such that $V \cap [x]_{\equiv}$ consists exactly of one element (axiom of choice). Then U can be written as disjoint union of the open balls $B(x, \rho(x))$, $x \in V$. In the case where U is a ρ-set we deal with the equivalence relation $x \ \equiv_\rho \ y$ iff $d(x, y) < \rho$ instead of $\equiv$. □

Lemma A.8. Let M be an ultrametric space and E an evaluation on M. Then, for each open subset U of M, whenever $U \ = \ \bigcup_{i \in I} \ B_i$ where $(B_i)_{i \in I}$ is a family of pairwise disjoint open balls then for each $\varepsilon > 0$ there exists a finite subset J of I with

$$E(U) \ - \ \varepsilon \ \leq \ \sum_{j \in J} \ E(B_j)$$

Proof. If $J \subseteq I$ is finite then $B_J \ \stackrel{\text{def}}{=} \ \bigcup_{j \in J} B_j$. Then, $E(B_J) \ = \ \sum_{j \in J} E(B_j)$. Let $\varepsilon > 0$. We show that there exists a finite subset J of I with $E(B_J) \geq E(U) - \varepsilon$. Let K be the set of finite subsets of I. Then, the family $(B_J)_{J \in K}$ is directed and $\bigcup_{J \in K} \ B_J \ = \ U$. Hence:

$$E(U) \ = \ \sup_{J \in K} \ E(B_J).$$

In particular, there exists a finite subset J of I with $E(B_J) \geq E(U) - \varepsilon$. □

Immediately by Lemma A.8 we obtain the following.

Corollary A.9. Each evaluation on an ultrametric space is uniquely determined by its value on the open balls.

Lemma A.10. Let M be an ultrametric space, E an evaluation on M and $0 < r \leq 1$. Then, the set of open balls $B = B(x, r)$ for some $x \in M$ with $E(B) \neq 0$ is countable.

I.e. there exists a countable subset N of M such that $E(B(x,r)) \neq 0$ implies $d(x,y) < r$ for some $y \in N$.

Proof. Since $M = \bigcup_{B \in I} B$ where I is the set of open balls $B = B(x,r)$ for some $x \in M$ and since $B \cap B' = \emptyset$ for all B, $B' \in I$, $B \neq B'$ we get by Lemma A.8: There exists finite subsets I_n of I with

$$1 - \frac{1}{2^n} \leq \sum_{B \in I_n} E(B).$$

W.l.o.g. $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ (otherwise we deal with $I'_n = I_0 \cup I_1 \cup \ldots \cup I_n$ instead of I_n). Let $J = \bigcup I_n$. Then, J is countable. Let $B \in I \setminus J$. We suppose $E(B) > 0$. Let n be a natural number with $1/2^n < E(B)$. Then,

$$1 = E(M) \geq E(B) + \sum_{B' \in I_n} E(B') \geq E(B) + 1 - \frac{1}{2^n} > 1.$$

Contradiction. Hence, $E(B) = 0$ for all $B \in I \setminus J$. □

In the next lemma we give sufficient – and by Lemma A.8 – necessary conditions for the extension of a given function $F : \mathcal{B} \to [0,1]$ to an evaluation.

Lemma A.11. Let M be an ultrametric space and $F : \mathcal{B} \to [0,1]$ a function which satisfies:

1. $F(M) = 1$
2. If $B_1, \ldots, B_n$ are pairwise disjoint open balls which are contained in some open ball B then
$$\sum_{i=1}^{n} F(B_i) \leq F(B).$$
3. Whenever B is an open ball and $B = \bigcup_{i \in I} B_i$ where $(B_i)_{i \in I}$ is a family of pairwise disjoint open balls then for each $\varepsilon > 0$ there exists a finite subset J of I with
$$F(B) - \varepsilon \leq \sum_{j \in J} F(B_j).$$

Then, there exists a unique evaluation E on M with $E(B) = F(B)$ for all $B \in \mathcal{B}$.

Proof. By Corollary A.9 there exists at most one evaluation E on M which extends F. We define E as follows:

$$E(U) \stackrel{\text{def}}{=} \sup \left\{ \sum_{B \in I} F(B) \;:\; I \in \mathcal{I}(U) \right\}$$

where $\mathcal{I}(U)$ denotes the collection of all finite sets consisting of pairwise disjoint open balls $B \subseteq U$. By Lemma A.7, $\mathcal{I}(U)$ is nonempty whenever $U \neq \emptyset$. Whenever $I \in \mathcal{I}(M)$ we put

$$F_I \stackrel{\text{def}}{=} \sum_{B \in I} F(B).$$

We show that E is an evaluation: we have $E(M) = 1$ since $I = \{M\} \in \mathcal{I}(M)$ and $F(M) = 1$. The monotonicity of E is clear since whenever $U \subseteq V$ then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$. Let $U, V \subseteq M$ be nonempty opens. We show that $E(U \cap V) + E(U \cup V) = E(U) + E(V)$.

Step 1: We show that $E(U \cap V) + E(U \cup V) \ \leq \ E(U) + E(V)$. Let $\varepsilon > 0$. We first show that there exists $I_U \in \mathcal{I}(U)$ and $I_V \in \mathcal{I}(V)$ with $E(U \cup V) + E(U \cap V) - \varepsilon \ \leq \ F_{I_U} + F_{I_V}$. (Then, we may conclude that $E(U) + E(V) \ \geq \ E(U \cup V) + E(U \cap V) - \varepsilon$ for all $\varepsilon > 0$. Hence, $E(U) + E(V) \geq E(U \cup V) + E(U \cap V)$.)

Let $J \in \mathcal{I}(U \cup V)$, $K \in \mathcal{I}(U \cap V)$ with $F_J \ \geq \ E(U \cup V) - \frac{1}{4} \cdot \varepsilon$ and $F_K \ \geq \ E(U \cap V) - \frac{1}{4} \cdot \varepsilon$. Then,

(*) $\quad F_J + F_K \ \geq \ E(U \cup V) + E(U \cap V) - \frac{1}{2} \cdot \varepsilon$

To prove the above we claim that each ball $B \in J$ can be written as disjoint union of open balls C satisfying $C \subseteq U$ or $C \subseteq V$.

Let $B \in J$. For each $x \in B$ we put:

$$r(x) \ \stackrel{\text{def}}{=} \ \begin{cases} \sup \{r > 0 : B(x,r) \subseteq B \cap U\} & : \ \text{if } x \in B \cap U, \\ \sup \{r > 0 : B(x,r) \subseteq B \cap V\} & : \ \text{if } x \in B \cap (V \setminus U). \end{cases}$$

Then, $r(x) > 0$ for all $x \in B$. We put $B_x \stackrel{\text{def}}{=} B(x, r(x))$. Then, either $B_x \subseteq U$ or $B_x \subseteq V$. Let X be the set of elements $x \in B \cap U$ with $B_x \subseteq B_y$ for some $y \in B \cap (V \setminus U)$. We can deal with the set of balls $C = B_x$ where $x \in V$ or $x \in U \setminus X$. (Note that for all x, $y \in \ B \cap (V \cup (U \setminus X))$ either $B_x = B_y$ or $B_x \cap B_y = \emptyset$.) Thus, we have shown that the claim is true.

Let $|J|$ be the cardinality of J and let $B \in J$. By assumption there exists a finite set I_B consisting of pairwise disjoint open balls $C \subseteq U$ or $C \subseteq V$ with

$$F(B) - \frac{1}{2|J|} \cdot \varepsilon \ \leq \ F_{I_B}.$$

Let J' be the set of all balls $C \in I_B$, $B \in J$. Then, J' is finite and

$$F_J \ - \ \frac{1}{2} \cdot \varepsilon \ = \ \sum_{B \in J'} F(B) \ - \ \frac{1}{2} \cdot \varepsilon \ \ \leq \ \ \sum_{B \in J'} \sum_{C \in I_B} F(C).$$

We put:

$$\begin{array}{lcl} J'_U & \stackrel{\text{def}}{=} & \{B \in J' : B \subseteq U\}, \ \ J'_V \ \stackrel{\text{def}}{=} \ J' \setminus J'_U \\ K_U & \stackrel{\text{def}}{=} & \{ \ C' \in K \ : \ C \cap B = \emptyset \ \forall \ B \in J'_U \ \}, \ \ K_V \ \stackrel{\text{def}}{=} \ K \setminus K_U \\ I_U & \stackrel{\text{def}}{=} & J'_U \cup K_U, \ \ I_V \ \stackrel{\text{def}}{=} \ J'_V \cup K_V \end{array}$$

Then, $I_U \in \mathcal{I}(U)$. We show $I_V \in \mathcal{I}(V)$. It is clear that all balls $B \in I_V$ are contained in V and that the balls of J'_V (and the balls of K_V) are pairwise disjoint. Suppose there are balls $B \in J'_V$ and $C \in K_V$ with $B \cap C \neq \emptyset$. Then, either $B \subseteq C$ or $C \subseteq B$. The first case is impossible since $B \not\subseteq U$ (by definition of J'_V) and $C \subseteq U \cap V$. The second case is impossible since then $C \cap B' \neq \emptyset$ for some $B \in J'_U$ and hence either $B' \subseteq C \subseteq B$ or $C \subseteq B \cap B'$ (which contradicts the assumption that the balls B, B' are disjoint). We obtain

$$F_{I_U} + F_{I_V} \;=\; \sum_{B \in J'} \sum_{C \in I_B} F(C) \;+\; F_K$$

$$\geq\; F_J - \frac{1}{2} \cdot \varepsilon \;+\; F_K \;\geq\; E(U \cup V) + E(U \cap V) \;-\; \varepsilon.$$

Step 2: We show that $E(U \cap V) + E(U \cup V) \;\geq\; E(U) + E(V)$. Let $\varepsilon > 0$ and let $I_U \in \mathcal{I}(U)$, $I_V \in \mathcal{I}(V)$ such that

$$F_{I_U} \;\geq\; E(U) - \frac{1}{2} \cdot \varepsilon, \quad F_{I_V} \;\geq\; E(V) - \frac{1}{2} \cdot \varepsilon.$$

Then, $K \stackrel{\text{def}}{=} \{B \cap C : B \in I_U, C \in I_V, B \cap C \neq \emptyset\}$ is a finite set of disjoint open balls which are contained in $U \cap V$. Let J be the set consisting of the following balls:

— $B \cup C$ where $B \in I_U$, $C \in I_V$, $B \cap C \neq \emptyset$
— $B \in I_U$ where $B \cap C = \emptyset$ for all $C \in I_V$
— $B \in I_V$ where $B \cap C = \emptyset$ for all $B \in I_U$

J is a finite set of pairwise disjoint open balls contained in $U \cup V$. (Note that whenever B, C are open balls with $B \cap C \neq \emptyset$ then either $B \subseteq C$ or $C \subseteq B$.) It is easy to see that $F_K + F_J = F_{I_U} + F_{I_V}$. Hence,

$$E(U \cup V) + E(U \cap V) \;\geq\; F_K + F_J \;=\; F_{I_U} + F_{I_V} \;\geq\; E(U) + E(V) - \varepsilon$$

for all $\varepsilon > 0$. Therefore, $E(U \cup V) + E(U \cap V) \;\geq\; E(U) + E(V)$.

Step 3: We show that E is continuous. Let U be a nonempty open set and let $(U_i)_{i \in I}$ be a directed family of open sets with $U = \bigcup U_i$. Since $U_i \subseteq U$ we have $E(U_i) \leq E(U)$ and therefore $\sup E(U_i) \;\leq\; E(U)$. For each $x \in U$ and $i \in I$ we put $r_i(x) \stackrel{\text{def}}{=} 0$ if $x \notin U_i$ and

$$r_i(x) \;\stackrel{\text{def}}{=}\; \sup\,\{r > 0 : B(x,r) \subseteq U_i\}$$

if $x \in U_i$. Let $r(x) \;\stackrel{\text{def}}{=}\; \sup_{i \in I}\; r_i(x)$. Then, $r_i(x) > 0$ and $B_x \;\stackrel{\text{def}}{=}\; B(x, r(x)) \;\subseteq\; U$. We define an equivalence relation $\equiv$ on U by $x \;\equiv\; y$ iff $B_x = B_y$. It is easy to see that $x \;\not\equiv\; y$ iff $B_x \cap B_y = \emptyset$ and that B_x is the equivalence class of x. For each equivalence class A, we define $B_A \stackrel{\text{def}}{=} B_x$ and $r_A \stackrel{\text{def}}{=} r(x)$, where x is a representative of A. We choose a real number ρ_A with $0 < \rho_A < r_A$. Then, U is the disjoint union of the balls B_A where A ranges over all equivalence classes. Let $\mathcal{C}_A$ be the set of all balls $B(x, \rho_A)$ where $x \in A$. B_A is the disjoint union of the balls $C \in \mathcal{C}_A$. For each ball $C \in \mathcal{C}_A$ there exists $i \in I$ and $x \in C$ such that $\rho_A < r_i(x) \leq r_A$. (Here, we use the fact that $x \equiv y$ implies $r_i(x) = r_i(y)$ for all $i \in I$.) Then, $C \subseteq\; B(x, r_i(x)) \subseteq U_i$. Hence, U is the disjoint union of the balls $C \in \mathcal{C}_A$ where A ranges over the equivalence classes. By definition of E there exists a finite set J of open balls where each ball $C \in J$ is contained in some of the sets U_i and which satisfies:

$$E(U) - \varepsilon \;\leq\; \sum_{C \in J} F(C)$$

Since $(U_i)_{i \in I}$ is directed and since J is finite there exists an index $i \in I$ with $C \subseteq U_i$ for all $C \in J$. Then, $\bigcup_{C \in J} C \subseteq U_i$. Thus, $E(U_i) \geq \sum_{C \in J} F(C) \geq E(U) - \varepsilon$. $\square$

Theorem A.12. (cf. Theorem 4.7) If M is an ultrametric space then $\mathcal{E}_1(M)$ is an ultrametric space. If M is complete then also $\mathcal{E}_1(M)$ is complete.

Proof. It is clear that the distance d on $\mathcal{E}_1(M)$ is a pseudo-ultrametric. If $d(E_1, E_2) = 0$ we have to show that $E_1(U) = E_2(U)$ for all opens U. Since $d(E_1, E_2) = 0$ we have $E_1(B) = E_2(B)$ for all $B \in \mathcal{B}$. U can be written as disjoint union of open balls: $U = \bigcup_{i \in I} B_i$. The family of sets $U_J = \bigcup_{i \in J} B_i$ where J is a finite subset of I is directed. For each finite subset J we have:

$$E_1(U_J) \;=\; \sum_{i \in J} E_1(B_i) \;=\; \sum_{i \in J} E_2(B_i) \;=\; E_2(U_J).$$

Since $U = \bigcup_J U_J$ we get $E_1(U) = \sup_J E_1(U_J) = \sup_J E_2(U_j) = E_2(U)$. We conclude that $\mathcal{E}_1(M)$ is an ultrametric space.

Now we suppose M to be complete. We show that $\mathcal{E}_1(M)$ is complete. Let $(E_n)_{n \geq 0}$ be a Cauchy sequence of evaluations on M. W.l.o.g. $d(E_n, E_N) \leq 1/2^{N+1}$ for all $0 \leq N \leq n$. For each open ball B, $(E_n(B))_{n \geq 0}$ is a Cauchy sequence. Note that the sequence $(E_n(B))$ is eventually constant as, for $B \in \mathcal{B}_\rho$, $E_n(B) = E_N(B)$ for all $n \geq N$ where $1/2^N \leq \rho$. We define a function $F : \mathcal{B} \to [0, 1]$ satisfying the conditions of Lemma A.11 as follows:

$$F(B) \stackrel{\text{def}}{=} \lim_{n \to \infty} E_n(B)$$

Then, $F(B) = E_n(B)$ for all $n \geq N$ and $B \in \mathcal{B}_{\frac{1}{2^N}}$. We show that F satisfies the conditions of Lemma A.11.

1. It is clear that $F(M) = 1$.
2. Let B be an open ball and $B_1, \ldots, B_n$ be disjoint open balls with $B_1 \cup \ldots \cup B_n \subseteq B$. We choose some $\rho > 0$ such that B, $B_i \in \mathcal{B}_\rho$, $i = 1, \ldots, n$, and some natural number N with $1/2^N < \rho$. Then,
$$F(B) \;=\; E_N(B) \;\geq\; \sum_{i=1}^{n} E_N(B_i) \;=\; \sum_{i=1}^{n} F(B_i).$$
3. Let $B \in \mathcal{B}_\rho$ and $(B_i)_{i \in I}$ a family of disjoint open balls B_i with $\bigcup B_i = B$. Let $\varepsilon > 0$. We choose a natural number N with $1/2^N < \rho$. Then, $F(B) = E_n(B)$ for all $n \geq N$. Because of Lemma A.8 there exist finite subsets I'_n of I with
$$\sum_{i \in I'_n} E_n(B_i) \;\geq\; F(B) - \frac{1}{2} \cdot \varepsilon.$$
We put $I_n = I'_N \cup I'_{N+1} \cup \ldots \cup I'_n$. Then, I_n are finite subsets of I with $I_N \subseteq I_{N+1} \subseteq \ldots$ and
$$a_{k,n} \stackrel{\text{def}}{=} \sum_{i \in I_k} E_n(B_i) \;\geq \sum_{i \in I'_n} E_n(B_i) \;\geq\; F(B) - \frac{1}{2} \cdot \varepsilon$$
for all $k \geq n$. Since $a_{k,n} \;\leq\; a_{k+1,n} \;\leq\; \ldots \;\leq\; E_n(B) \;=\; F(B)$ the limit $\lim_{k \to \infty} a_{k,n}$ exists. With
$$a_n \stackrel{\text{def}}{=} \lim_{k \to \infty} a_{k,n}$$

we have: $a_n \;\geq\; a_{n,n} \;\geq\; F(B) - \frac{1}{2} \cdot \varepsilon$. Since I_k is finite we may choose some $\rho_k > 0$ such that $B_i \in \mathcal{B}_{\rho_k}$ for all $i \in I_k$. For all $n \geq N$ with $1/2^n < \rho_k$, $E_n(B_i) \;=\; F(B_i)$ for all $i \in I_k$. Thus,

$$a_{k,n} \;=\; \sum_{i \in I_k} E_n(B_i) \;=\; \sum_{i \in I_k} F(B_i) \;\stackrel{\text{def}}{=}\; A_k.$$

Then, $A_k \;\leq\; A_{k+1} \;\leq\; \ldots \;\leq\; F(B)$ (since $(B_i)_{i \in I_k}$ is a finite family of pairwise disjoint open balls in B). Hence, $\lim_{k \to \infty} A_k$ exists, i.e.

$$\sum_{i \in K} F(B_i) \;\text{ where }\; K \;\stackrel{\text{def}}{=}\; \bigcup_{n \geq N} I_n$$

is convergent. Moreover,

$$\sum_{i \in K} F(B_i) \;=\; \lim_{k \to \infty} \lim_{n \to \infty} a_{k,n} \;=\; \lim_{n \to \infty} \lim_{k \to \infty} a_{k,n} \;=\; \lim_{n \to \infty} a_n \;\geq\; F(B) - \frac{1}{2} \cdot \varepsilon.$$

Hence, there exists a finite subset J of K with $\sum_{j \in J} F(B_j) \;\geq\; F(B) - \varepsilon$.

We conclude that F satisfies the conditions of Lemma A.11. Hence, there exists a unique evaluation E on M which extends F. Then, $E(B) = F(B) = E_N(B)$ for all $B \in \mathcal{B}_\rho$ where $\rho > 1/2^N$. Therefore, $d(E, E_N) \leq 1/2^N$ and $E = \lim E_n$. $\square$

Recall that, for M to be a complete ultrametric space, the function $\mathcal{D}_1(M) \to \mathcal{E}_1(M)$, $\mu \mapsto E_\mu$, is injective. Thus, $\mathcal{D}_1(M)$ can be viewed as a subspace of $\mathcal{E}_1(M)$.

Theorem A.13. (cf. Theorem 4.7) Let M be a complete ultrametric space. Then, $\mathcal{E}_1(M)$ is the completion of $\mathcal{D}_1(M)$.

Proof. We have to show that $\{E_\mu : \mu \in \mathcal{D}_1(M)\}$ is a dense subspace of $\mathcal{E}_1(M)$. Let $E \in \mathcal{E}_1(M)$. For all $n \geq 0$, there exists a countable subset N_n of M such that $E(B(x, 1/2^n)) \neq 0$ implies $d(x, y) < 1/2^n$ for some $y \in N_n$ (Lemma A.10). For each ball $B \in \mathcal{B}_{1/2^n}$ we choose an element $x_B \in B$ and put

$$\mu_n(y) \;\stackrel{\text{def}}{=}\; \begin{cases} E(B) & : \text{ if } y = x_B \text{ for some } B \in \mathcal{B}_{1/2^n} \\ 0 & : \text{ otherwise.} \end{cases}$$

Then, μ_n is a distribution on M and $d(E_{\mu_n}, E) \;\leq\; 1/2^n$. Hence, $E = \lim E_{\mu_n}$. $\square$

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