On duality for the modal μ-calculus

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Abstract. We consider the modal μ-calculus due to Kozen, which is a finitary modal logic with least and greatest fixed points of monotone operators. We extend the existing duality between the category of modal algebras with homomorphisms and the category of descriptive modal frames with contractions to the case of having fixed point operators. As a corollary, we obtain a completeness result for Kozen's original system with respect to a certain class of modal frames.

1 Introduction

Modal action logic provides a framework for reasoning about the finitary properties of transition systems. In this paper we consider an enhancement of such a logic with fixed point operators to allow infinitary properties such as fairness (for earlier work, see for example [13] and [10])to be expressed. Fairness properties are statements of substantial complexity. Intuitively, (strong) fairness can be expressed as $\forall P$. *P infinitely often enabled implies P infinitely often taken*, where P is a process (we refer the reader to *e.g.* [2, 9] for more detail). As a concrete system which is capable of expressing fairness, we adopt the modal μ-calculus due to Kozen [7]. It is essentially a finitary, propositional modal logic with least and greatest fixed points of monotone operators. Models for the calculus are labelled transition systems; the closed formulas describe sets of states, and the open formulas can be viewed as maps on the powerset of states. These maps are monotone but not necessarily order-continuous because there exist models and formulas involving fixed points whose interpretation in these models requires unfoldings for ordinals higher than ω. The appealing features of the modal μ-calculus are: its decidability (which should be contrasted with the Hennessy-Milner logic), and the expressive power (for example, it is strictly more expressive than PDL and CTL*). To give some intuition, the following modal μ-calculus formulas have the corresponding meaning in any model:

1. $\mu X.[a]X$ (every a-path is finite);
2. $\nu X.\langle a\rangle X$ (there exists an infinite a-path);
3. $\mu X.\varphi \vee [a]X$ (along every a-path eventually φ holds – liveness);

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4. $\nu X.\langle a\rangle \mathbf{t} \wedge [a]X$ (a is always possible – safety);
5. $\nu Z.\mu Y.(\langle a\rangle \mathbf{t} \vee [L]Y) \wedge [L]Z \wedge \langle L\rangle \mathbf{t}$ (a is possible infinitely often);
6. $\nu Z.\mu Y.[L-a]Y \wedge [L]Z \wedge \langle L\rangle \mathbf{t}$ (a happens infinitely often)

where L denotes some (finite) set of actions.

One difficulty with the modal μ-calculus is the lack of a satisfactory completeness statement (satisfactory in the sense of being sound over extensions to the logic) over all models, although we should point out that a number of weaker, but workable, systems are in existence, *e.g.* [7, 16, 18]. Although known to be decidable, see *e.g.* [8], the modal μ-calculus has not lent itself to an axiomatic characterization. The original calculus introduced by Kozen [7] is not yet known to be complete, although the subcalculus of *aconjunctive* formulas is shown to be so in [7]. Kozen obtains a completeness result in [8] by introducing an infinitary proof rule. Walukiewicz [18] obtains a similar result with the addition of an induction schema. Unfortunately in both cases the rules are only sound for validity rather than being sound in every model.

The aim of the paper is to cast light on this problem by deriving a suitable duality theory for the modal μ-calculus. Duality theory originates from Stone, who set out to formulate a representation theorem for Boolean algebras as a field of sets so that the operations of conjunction, disjunction and negation can be modelled by their set-theoretic counterparts. It was observed later that the representation theorem fitted into the more general categorical framework of a duality between Boolean algebras and Stone spaces - for more detail see [6]. We take an existing duality for modal algebras [14], and extend it to the case of having fixed point operators.

2 Notation and preliminaries

Some familiarity with Boolean algebras, modal logics and basic category theory is assumed.

Let (X, T) be a topological space, where X is a set of points and T is a basis for a topology. (X, T) is *totally disconnected* iff the only connected subsets are the singleton sets ($Y \subseteq X$ is *connected* iff the only clopen sets in the subspace topology are $\emptyset$ and Y). (X, T) is *totally separated* iff every pair of distinct points can be separated by a clopen subset, *i.e.* given x, y there exists a clopen set C such that $x \in C$ and $y \notin C$. A space is a *Stone space* iff it is compact, Hausdorff and totally disconnected, or, equivalently, compact and totally separated.

By a *field of subsets* we shall mean a family of subsets closed under the set-theoretic operations of intersection, union and complement.

The *Stone space* of a Boolean algebra B is defined by taking the set of all ultrafilters $U(B)$ as points, and the family $\beta(b) = \{U \in U(B) \mid b \in U\}$ as the clopen basis.

We recall two well-known characterizations of least and greatest fixed-points of monotone functions on a complete lattice. For an overview of fixed point theorems see *e.g.* [11]. The following characterization is as the meet or join of the set of *pre-fixed-points*.

Lemma 1. (Attributed to Knaster & Tarski) *Let $(L, \leq)$ be a complete lattice. Then every monotone function $f : L \longrightarrow L$ has the least fixed point:*

$$\mu x.f(x) = \bigwedge \{a \in L \mid f(a) \leq a\}$$

and its dual, the greatest fixed point:

$$\nu x.f(x) = \bigvee \{a \in L \mid a \leq f(a)\}.$$

This first characterization is often criticized for being non-constructive, which has prompted several authors to formulate the following alternative characterization.

Lemma 2. (Attributed to Hitchcock & Park) *Let $(L, \leq)$ be a complete lattice, $f : L \longrightarrow L$ a monotone function. Define the (increasing) chain of unfoldings of f by:*

$$\begin{aligned} f^0(u) &= u \\ f^{\alpha+1}(u) &= f(f^\alpha(u)) && \alpha \text{ successor ordinal} \\ f^\lambda(u) &= \bigvee \{f^\alpha(u) \mid \alpha < \lambda\} && \lambda \text{ limit ordinal.} \end{aligned}$$

Then there exists an ordinal α of cardinality less than or equal to that of L such that $\beta \geq \alpha$ implies

$$\mu x.f(x) = f^\beta(0).$$

Dually, define the (decreasing) chain of unfoldings of f by:

$$\begin{aligned} f_0(u) &= u \\ f_{\alpha+1}(u) &= f(f_\alpha(u)) && \alpha \text{ successor ordinal} \\ f_\lambda(u) &= \bigwedge \{f_\alpha(u) \mid \alpha < \lambda\} && \lambda \text{ limit ordinal.} \end{aligned}$$

Then there exists an ordinal α of cardinality less than or equal to that of L such that $\beta \geq \alpha$ implies

$$\nu x.f(x) = f_\beta(1).$$

It should be noted that these two characterizations coincide on complete lattices, but not, in general, on weaker structures. Of course, showing the existence of the ordinal α requires the axiom of choice so it is not clear whether this formulation is really any more constructive.

3 Modal μ-calculus

We take Act to be a fixed finite set of *actions*. The set T_μ of terms of the modal μ-calculus is defined inductively as follows. Suppose we are given a countable set of variables $V = \{x_0, x_1, \ldots\}$. The *μ-terms* are:

$V \subseteq T_\mu$	(variables)
$\underline{0} \in T_\mu$	(constants)
$t, t' \in T_\mu \Rightarrow t \wedge t', \neg t, [a]t \in T_\mu$	$a \in \mathsf{Act}$
$t \in T_\mu,\ x \in V,\ x$ positive in $t \Rightarrow \mu x.t \in T_\mu$	

where x *is positive in* t is a syntactic restriction defined as x occurring within an even scope of negations. This restriction implies monotonicity when viewing (open) terms as maps on formulas. The connectives $\vee$ and $\langle a \rangle$ are the usual duals to $\wedge$ and $[a]$ respectively. In particular,

$$\langle a \rangle t =_{\text{def}} \neg [a] \neg t.$$

The variable-binding operator μ denotes the least fixed point. The greatest fixed point ν is defined as the dual of the least fixed point as follows:

$$\nu x.t =_{\text{def}} \neg \mu y. \neg t(x := \neg y).$$

We give a sequent style presentation of the system due to Kozen [7]. The structural rules and rules for the basic connectives are those for Gentzen's system **LK** for classical logic [3]. We augment these with the rules for minimal modal action logic:

$$\frac{\{x \mid [a]x \in \Gamma\}, \varphi \vdash \{y \mid \langle a \rangle y \in \Delta\}}{\Gamma, \langle a \rangle \varphi \vdash \Delta} (\Diamond L) \quad \frac{\{x \mid [a]x \in \Gamma\} \vdash \varphi, \{y \mid \langle a \rangle y \in \Delta\}}{\Gamma \vdash [a]\varphi, \Delta} (\Box R)$$

(the latter being the dual of the former), together with the following rules for the μ- and ν-terms (strictly speaking, since ν is a dual of μ, the rules for ν are not necessary; we include them here for the sake of clarity):

$$\frac{\varphi(x := \psi) \vdash \psi}{\mu x.\varphi \vdash \psi} (\mu L) \; x \text{ not free in } \psi \qquad \frac{\Gamma \vdash \varphi(x := \mu x.\varphi), \Delta}{\Gamma \vdash \mu x.\varphi, \Delta} (\mu R)$$

$$\frac{\psi \vdash \varphi(x := \psi)}{\psi \vdash \nu x.\varphi} (\nu L) \; x \text{ not free in } \psi \qquad \frac{\Gamma, \varphi(x := \nu x.\varphi) \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta} (\nu R)$$

In the light of the two characterizations of fixed points mentioned in Section 2, the rule (μL) can be explained as stating that the term $\mu x.\varphi$ is a *lower bound* for pre-fixed-point terms, and the rule (μR) can be read as stating $\mu x.\varphi$ is a *pre-fixed-point* of the map φ.

4 Modal algebras, modal frames and completeness

We use the framework of modal duality as originally proposed by Halmos [5] and later developed by Sambin and Vaccaro [14] and extend this to give an account of the fixed point operators.

Given any modal logic L with simple modalities $\Box$ and $\Diamond$ satisfying the analogue of rules $(\Diamond L)$ and $(\Box R)$ above, one can give (algebraic) semantics to L in terms of a *modal algebra* as defined below.

Definition 3. A *modal algebra* [14] is a pair $M = \langle B, \tau \rangle$ where B is a Boolean algebra and τ is an operation satisfying:

(A1) $\tau 1 = 1$
(A2) $\tau(a \wedge b) = \tau a \wedge \tau b$.

One can now state completeness of algebraic semantics as follows:

> "for every logic L and every formula φ, $\vdash_L \varphi$ iff φ is valid in every modal algebra satisfying the axioms of L".

The alternative, relational, semantics for the logic L is based on the notion of a Kripke frame, that is, a pair $\langle X, r\rangle$ where X is a set of possible worlds and r is a binary relation of accessibility between worlds. Propositions are modelled by sets of possible worlds (subsets of X), the classical connectives are modelled by their set-theoretic counterparts and the $\Box$-modality is modelled by the operation r^* defined by

$$r^*C = \{x \in X \mid \forall y.xry \Rightarrow y \in C\}.$$

Unfortunately, as has been shown by Fine [1] and Thomason [17], the statement of completeness of such semantics, which is expressed as the statement:

> "for every logic L, $\vdash_L \varphi$ iff φ is valid in every Kripke frame in which every axiom of L is valid"

fails for some logics. The remedy is to extend the notion of a frame with additional information to give rise to a *modal frame*, that is, a triple $\langle X, r, T\rangle$ where $\langle X, r\rangle$ is a Kripke frame and T is a field of subsets of X (hence a Boolean algebra) closed under the operation r^*.

Definition 4. [14] A *modal frame* is a triple $F = \langle X, r, T\rangle$ where $\langle X, r\rangle$ is a Kripke frame and T is a field of subsets of X closed under the operation r^* defined by

$$r^*C = \{x \in X \mid \forall y.xry \Rightarrow y \in C\}.$$

The field of sets T serves a dual purpose: on the one hand, it is a modal algebra of *basic propositions* hiding inside a frame, while, on the other, it is a *clopen basis* for a topology on X. Now, enlarging the class of models in this way allows a completeness result to be proved for an arbitrary modal logic, see *e.g.* [14].

It should be noted that, given a Kripke frame $\langle X, r\rangle$, the powerset $\wp(X)$ can always serve as the field of subsets closed under r^* so every Kripke frame is a modal frame in which the basic propositions are allowed to range over *all* subsets of X, as opposed to just a *subalgebra* T. The idea of restricting the propositions to a subalgebra corresponds to the intuition that only some of the potential propositions on X correspond to *observable* properties of the transition system.

Observe that Kozen's original models are essentially Kripke frames, hence the interpretation of fixed points is in the complete lattice $(\wp(X), \subseteq)$, and not in the field of subsets T. The argument used here should not be construed as an argument for incompleteness of Kozen's system, but only as an explanation of the differences between the results presented here and that of [7].

5 Modal μ-algebras

We generalize the notion of a modal algebra to allow an indexed family of modalities. Consequently, we consider a *modal algebra* to be a pair $M = \langle B, \{\tau_a\}_{a \in \mathsf{Act}} \rangle$ where B is a Boolean algebra and $\{\tau_a\}$ is a family of operations on B satisfying $(A1)$ and $(A2)$ above. Following some ideas due to Niwinski [12], we now extend the notion of a modal algebra by adding fixed point operators as follows.

Let $M = \langle B, \{\tau_a\} \rangle$ be a modal algebra. An *assignment* $A : V \longrightarrow M$ is a map from variables V to the elements of the modal algebra M. We define for each assignment $A : V \longrightarrow M$ a partial function $[\![\cdot]\!]_A : T_\mu \to M$ as follows:

$$
\begin{array}{l}
[\![x]\!]_A = A(x) \\
[\![\underline{0}]\!]_A = 0 \\
[\![t \wedge t']\!]_A = [\![t]\!]_A \wedge [\![t']\!]_A \\
[\![\neg t]\!]_A = \neg [\![t]\!]_A \\
[\![[a]t]\!]_A = \tau_a [\![t]\!]_A \\
[\![\mu x.t]\!]_A = \mathrm{lfp}(\lambda b.[\![t]\!]_{A(x:=b)}) \text{ if this exists, undefined otherwise}
\end{array}
$$

where $\mathrm{lfp}(\lambda b.f(b))$ denotes the least fixed point of the operator $\lambda b.f(b)$ on M.

Definition 5. *(Modal μ-algebra.)* Let M be a modal algebra. Then M is a *modal μ-algebra* if, for all assigments A and terms t, $[\![t]\!]_A$ is defined.

There are two important special cases of modal μ-algebras corresponding to the two characterizations of least fixed points given in Section 2. We shall say that a modal μ-algebra M is of *the first kind* if, for all terms t and assignments A, the meet

$$\bigwedge \{ b \in M \mid [\![t]\!]_{A(x:=b)} \leq b \}$$

of pre-fixed-points exists. Note that if it exists, then this meet is the least fixed point of the operator $\lambda b.[\![t]\!]_{A(x:=b)}$, but that the least fixed point might exist without it being the meet of the pre-fixed-points.

Similarly, we shall say that a modal μ-algebra M is *of the second kind* if for all terms t, assignments A and ordinals α, the unfolding

$$(\lambda b.[\![t]\!]_{A(x:=b)})^\alpha(0)$$

exists. It follows that for some ordinal α of cardinality less than or equal to that of M, this unfolding is the least fixed point of the operator $\lambda b.[\![t]\!]_{A(x:=b)}$, but again, the least fixed point might exist without all the unfoldings above.

Complete modal algebras are, of course, modal μ-algebras of both the first and second kind. In general, a modal algebra might be of one kind or the other or not of either kind. We shall show that the Lindenbaum algebra of the proof system given in Section 3 is of the first kind, though it is not known whether it is of the second. We anticipate that algebras of the second type will correspond to a different proof system.

6 Modal μ-frames

We generalize the notion of a modal frame to allow an indexed family of accessibility relations. Thus, a *modal frame* is a triple $F = \langle X, \{r_a\}_{a \in \mathsf{Act}}, T \rangle$ where $\langle X, \{r_a\} \rangle$ is a transition system (the indexed version of a Kripke frame) and T is a field of subsets of X closed under the modality ${r_a}^*$, derived from the accessibility relation r_a, for each $a \in \mathsf{Act}$. The field T is, at the same time, the modal algebra of basic propositions, and a clopen basis for a topology on X. In order to recover a modal algebra from a modal frame $\langle X, \{r_a\}, T \rangle$, we simply take the field T together with the operations ${r_a}^*$. More formally, any modal frame $\langle X, \{r_a\}, T \rangle$ gives rise to a modal algebra $\langle T, \{{r_a}^*\} \rangle$.

The key element in our extension of duality theory is the idea that T might also be a modal μ-algebra:

Definition 6. A *modal μ-frame* is a modal frame $\langle X, \{r_a\}, T \rangle$ in which $\langle T, \{{r_a}^*\} \rangle$ is a modal μ-algebra.

We shall say that a modal μ-frame is of the first or second kind respectively if its corresponding algebra is of the first or second kind.

7 From modal μ-algebras to modal μ-frames

Let $M = \langle B, \{\tau_a\} \rangle$ be a modal μ-algebra. An established result [14] is that every modal algebra determines a modal frame. This frame is constructed by taking the Boolean ultrafilters $U(B)$ as the set of worlds, the relations τ_{a*} defined by:

$$S \tau_{a*} T \iff \forall b \in B.\, \tau_a(b) \in S \Rightarrow b \in T$$

as the accessibility relations, and the family $\beta(B) = \{\beta(b) \mid b \in B\}$ of sets of ultrafilters as the basic propositions:

$$\beta(b) = \{U(B) \mid b \in U\}.$$

Formally, the triple $\langle U(B), \{\tau_{a*}\}, \beta(B) \rangle$ is a modal frame, indeed the dual frame of M when we consider M purely as a modal algebra. We now propose to commence the programme of extending duality theory to the μ-construction by showing that $\langle U(B), \{\tau_{a*}\}, \beta(B) \rangle$ is a modal μ-frame. In other words, we need to show that $\langle \beta(B), \{(\tau_{a*})^*\} \rangle$ is a modal μ-algebra.

β gives rise to a mapping from assignments in M to assignments in $\langle \beta(B), \tau_{a*} \rangle$, defined by

$$\beta(A) = x \mapsto \beta(A(x)).$$

The following proposition shows that β commutes with the interpretation of terms.

Proposition 7. *For all $t \in T_\mu$, and assignments A,*

$$\beta([\![t]\!]_A) = [\![t]\!]_{\beta(A)}.$$

Proof. By induction on the structure of t. We know from [14] that β is an isomorphism of modal algebras, so in the base case, where t contains no fixed point operators, the result follows immediately from the definition of the interpretation function $[\![\cdot]\!]_A$.

Suppose now that the result holds for all terms t with n fixed point operators, and consider the term $\mu x.t$. If u is a fixed point of $\lambda b.[\![t]\!]_{A(x:=b)}$ then

$$\begin{aligned}\beta(u) &= \beta([\![t]\!]_{A(x:=u)}) \\ &= [\![t]\!]_{\beta(A(x:=u))} && \text{by induction hypothesis} \\ &= [\![t]\!]_{\beta A(x:=\beta(u))}\end{aligned}$$

so $\beta(u)$ is a fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$. The converse also holds because β is an isomorphism. Since β and β^{-1} are order preserving, we have that u is the *least* fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$ if and only if $\beta(u)$ is the *least* fixed point of $\lambda c.[\![t]\!]_{\beta A(x:=c)}$, and hence

$$\beta([\![\mu x.t]\!]_A) = [\![\mu x.t]\!]_{\beta(A)}.$$

□

We can now give the following representation theorem for modal μ-algebras extending the traditional Stone representation of Boolean algebras and the results of [4, 14].

Theorem 8. (Representation for modal μ-algebras) $\langle \beta(B), \{(\tau_{a*})^*\}\rangle$ *is a modal* μ*-algebra isomorphic to* $\langle B, \{\tau_a\}\rangle$.

Proof. Let t be any term and A be an arbitrary assignment in $\beta(B)$. We need to confirm that $[\![t]\!]_A$ is defined. Observe that A must be $\beta(A')$ for some assignment A' in M. Furthermore, $[\![t]\!]_{A'}$ must be defined since M is a modal μ-algebra. Hence

$$[\![t]\!]_A = [\![t]\!]_{\beta(A')} = \beta([\![t]\!]_{A'}) \qquad \text{Proposition 7}$$

Thus $[\![t]\!]_A = \beta([\![t]\!]_{A'})$ must be defined because β is a total function. So $\langle \beta(B), \{\tau_a\}\rangle$ is a modal μ-algebra, and furthermore, by Proposition 7,

$$\beta : \langle B, \{\tau_a\}\rangle \rightarrow \langle \beta(B), \{(\tau_{a*})^*\}\rangle$$

is an isomorphism of modal μ-algebras. □

Furthermore, it is straightforward to prove that if $\langle \beta(B), \{(\tau_{a*})^*\}\rangle$ is a modal μ-algebra of the first (second) kind then so is $\langle \beta(B), \{(\tau_{a*})^*\}\rangle$.

One outcome of the reprepresentation theorem is that we have arrived at an appropriate notion of *homomorphism* between modal μ-algebras since β should be an example of such. We note that it preserves the modal structure of the algebra as well as preserving the (interpretation of) fixed points, in the sense that the image of the interpretation a fixed point is the interpretation of the fixed point in the target algebra.

8 A completeness result

As a first step, we construct the Lindenbaum algebra of the proof system described in Section 3 and show that it is a modal μ-algebra of the first type. A completeness result then follows by using the representation theorem of the previous section.

Let $\simeq$ be the equivalence relation defined on propositions by $\varphi \simeq \psi$ if and only if $\vdash \varphi \iff \psi$, and let

$$[\varphi] = \{\psi \mid \varphi \simeq \psi\}$$
$$T_\mu/_\simeq = \{[\varphi] \mid \varphi \in T_\mu\}.$$

We can define Boolean operations on $T_\mu/_\simeq$ in the usual way and the resulting structure $B = (T_\mu/_\simeq, \wedge, \vee, \neg, 0, 1)$ is, of course, a Boolean algebra. If we add the operators τ_a defined by

$$\tau_a[\varphi] = [\,[a]\varphi\,]$$

we have a modal algebra $\mathcal{LA} = \langle B, \{\tau_a\}\rangle$, the *Lindenbaum algebra* for the modal μ-calculus.

We observe that in the Lindenbaum algebra the ordering $\leq$ on elements is defined by

$$[\varphi] \leq [\psi] \text{ iff } \varphi \vdash \psi.$$

Lemma 9. *Let φ be a term with free variables $u_1, u_2, \ldots, u_n$ and $A : V \longrightarrow T_\mu$ be an assignment mapping each of the variables u_i to an equivalence class $[\theta_i]$. Then*

$$[\![\varphi]\!]_A = [\varphi(u_1 := \theta_1, \ldots, u_n := \theta_n)]$$

provided that θ_i is free for u_i in φ.

Proof. Induction on the structure of φ. □

Proposition 10. *Let φ be a term with free variables $x, u_1, u_2, \ldots, u_n$ and $A : V \longrightarrow T_\mu$ be an assignment mapping each of the variables u_i to an equivalence class $[\theta_i]$. Then*

$$[\mu x.\varphi'] = \bigwedge \{[\psi] \mid [\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]\}$$

where $\varphi' = \varphi(u_1 := \theta_1, \ldots, u_n := \theta_n)$ provided θ_i is free for u_i in φ for all i, and ψ is free for x in φ'.

Proof. Consider any $[\psi]$ such that $[\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]$. Then, by the lemma above, we have that

$$[\varphi'(x := \psi)] = [\varphi(x := \psi, u_1 := \theta_1, \ldots, u_n := \theta_n)] \leq [\psi].$$

It follows that

$$\begin{aligned} \varphi'(x := \psi) \vdash \psi &\Rightarrow \mu x.\varphi' \vdash \psi \\ &\Rightarrow [\mu x.\varphi'] \leq [\psi]. \end{aligned}$$

Thus $\mu x.\varphi'$ is a lower bound of $\{[\psi] \mid [\![\varphi]\!]_{A(x:=[\psi])} \leq [\psi]\}$.

Furthermore

$$\begin{aligned}\varphi'(x := \mu x.\varphi) \vdash \mu x.\varphi' &\Rightarrow [\varphi'(x := \mu x.\varphi')] \leq [\mu x.\varphi'] \\ &\Rightarrow [\mu x.\varphi'] \in \{[\psi] \mid [\varphi'(x := \psi)] \leq [\psi]\}.\end{aligned}$$

Thus $[\mu x.\varphi']$ is the greatest lower bound. □

Theorem 11. *$\mathcal{LA}$ is a modal μ-algebra of the first kind.*

Proof. Immediate from the above. By α-conversion, we can always change names of bound variables so that the conditions of freeness are respected. □

As a modal μ-algebra the Lindenbaum algebra gives rise to a frame where, by our previous result, the field of sets is a modal μ-algebra of the first kind. This frame can serve as a canonical model for the logical system which we have described. Hence we have

Corollary 12. *Kozen's axiomatisation of the modal μ-calculus is complete with respect to the class of modal μ-frames of the first kind.*

9 Duality theory

In this section we derive a duality theory for modal μ-algebras. We consider the special cases of modal μ-algebras of the first and second kind separately.

9.1 Duality for modal algebras

We begin by reviewing the existing duality theory for modal algebras. This duality is based on the original Stone duality (see Section 1) for Boolean algebras, since modal algebras $\langle B, \{\tau_a\}_{a \in \mathsf{Act}} \rangle$ are Boolean algebras B with an indexed family of operators $\{\tau_a\}$ on B. Homomorphism of modal algebras are Boolean algebra morphisms which, in addition, preserve each τ_a.

Following Halmos [5] and Sambin and Vaccaro [14], we think of the category Mal of modal algebras and homomorphisms as obtained from a larger auxiliary category Bal of Boolean algebras and hemimorphisms (*i.e.* maps $\tau : B \longrightarrow B'$ preserving finite meets). A modal algebra $\langle B, \{\tau_a\} \rangle$ (an *object* in Mal) can be identified with an indexed family of arrows $B \stackrel{\tau_a}{\rightarrow} B$ in Bal, and the *arrows* of Mal are those morphisms h in Bal which make each of the following squares:

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ {\scriptstyle \tau_a}\downarrow & & \downarrow{\scriptstyle \tau'_a} \\ B & \xrightarrow[h]{} & B' \end{array}$$

commute.

The category Fra of (all) frames has as objects (indexed) modal frames $\langle X, \{r_a\}_{a \in \mathsf{Act}}, T\rangle$ and as arrows *weak contractions* c, that is, maps which are continuous in the topology on X induced by T satisfying $\mathrm{Cl}(r_a c x) = \mathrm{Cl}(c r_a x)$ for all $a \in \mathsf{Act}$, $x \in X'$. We would like to reproduce the view of modal algebras as "living" in a larger category for frames as well. Modal frames $\langle X, \{r_a\}, T\rangle$ can be viewed as indexed families of arrows $\langle X, T\rangle \stackrel{r_a}{\to} \langle X, T\rangle$ in the (larger) category Spa of *spaces* (where X denotes the set of points and T is a field of subsets of X serving as a clopen basis of the topology) with *continuous relations* as morphisms (that is, $r_a \subseteq X \times Y$ is a morphism from (X, T) to (Y, U) iff for every $D \in U$, ${r_a}^* D \in T$ where

$$ {r_a}^* D = \{x \in X \mid \forall y \in Y. x r_a y \Rightarrow y \in D\}). $$

Thus, arrows on frames are those weak contractions c which make each of the following diagrams commute:

$$\begin{array}{ccc} (X,T) & \xleftarrow{c} & (X',T') \\ \uparrow{\scriptstyle r_a} & & \uparrow{\scriptstyle r'_a} \\ (X,T) & \xleftarrow[c]{} & (X',T') \end{array}$$

There are functors $(\cdot)^*$ from Spa to Bal and $(\cdot)_*$ from Bal to Spa defined by:

$$\begin{aligned} (\langle B, \{\tau_a\}\rangle)_* &= (\langle U(B), \tau_{a*}, \beta(B)\rangle) \\ (\langle X, \{r_a\}, T\rangle)^* &= (\langle T, r_a^*\rangle) \end{aligned}$$

These functors give rise to an adjunction, which restricts to a duality for the subcategory DFra of descriptive frames. A modal frame $F = \langle X, \{r_a\}, T\rangle$ is said to be *refined* iff the following conditions hold:

$$\begin{aligned} &\text{(F1)}\ \forall a \in \mathsf{Act}.(x r_a y \iff \forall C \in T. x \in {r_a}^* C \Rightarrow y \in C) \\ &\text{(F2)}\ \forall C \in T.(x \in C \iff y \in C) \Rightarrow x = y \end{aligned}$$

In addition, if F satisfies:

$$\text{(F3)}\ \forall V \in \wp(T), V \text{ an ultrafilter}, \bigcap_{S \in V} S = \{x\} \text{ for some } x \in X$$

then it is called *descriptive* after Goldblatt [4]. Axiom (F1), which states that r_a is *point-closed*, guarantees that the modalities can be recovered from the relations r_a. Axiom (F2) can be explained as the *Hausdorff separation axiom*, since its meaning is that any two points with the same system of neighbourhoods are equal. Finally, axiom (F3) enforces *compactness*, since it states that every ultrafilter of neighbourhoods converges, and hence must have a non-empty intersection (which must be the singleton set $\{x\}$ by (F2)).

DFra is the category of descriptive frames with *contractions* (*i.e.* maps

$$c : \langle X', \{r'_a\}, T' \rangle \longrightarrow \langle X, \{r_a\}, T \rangle$$

such that c is continuous and satisfies the stronger condition $cr'_a x = r_a cx$ for every $a \in \mathsf{Act}$, $x \in X'$).

Theorem 13. [14] (Modal duality) *The categories* Mal *of modal algebras and homomorphisms, and* DFra *of descriptive frames and contractions, are dual to each other.*

9.2 Duality for Boolean algebras with infinite operations

Let B, B' be Boolean algebras, and let $h : B \longrightarrow B'$ be a Boolean algebra homomorphism. h is said to *preserve the join* $\bigvee_{i \in I} a_i$, where I is an index set, iff

$$h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i).$$

Let β be the isomorphism of a Boolean algebra B onto the clopen subsets of the Stone space $(U(B), \beta(B))$ of B, and let $a = \bigvee_{i \in I} a_i$. Since β is an isomorphism, the join of $\beta(a_i)$ must exist in the algebra of clopens, but it does not have to be the set-theoretic union (since, if this were the case, B would have been distributive). It can be shown [15] that for a Boolean algebra B the joins, if they exist, are the (topological) *closures* of certain unions in the Stone space of B, in the sense stated below.

Proposition 14. *Let β be the isomorphism of a Boolean algebra B onto the clopen subsets of the Stone space of B. For any element a, the following are equivalent:*

1. $a = \bigvee_{i \in I} a_i$
2. $\beta(a) = \mathrm{Cl}(\bigcup_{i \in I} \beta(a_i))$
3. *the* defect *set* $S = \beta(a) \setminus \bigcup_{i \in I} \beta(a_i)$ *is nowhere dense,* i.e. $\mathrm{Int}(\mathrm{Cl}(S)) = \emptyset$

where I is some indexed set.

The elements of the set S are all and only those ultrafilters which contain $\bigvee_{i \in I} a_i$ without containing any of the a_i. Such an ultrafilter, often called *non-standard*, is not κ-complete (where κ is the cardinality of I). So, if an infinite join exists, then the set-theoretic difference between its β-image and the union of the β-images of its approximants is nowhere dense, and cannot contain a β-image of an element of the Boolean algebra. We call such sets *defect sets* following Sikorski [15]. The case of infinite meets is dual.

Traditionally, Proposition 14 has been used to formulate a duality theorem between the category of κ-complete Boolean algebras with homorphisms preserving κ-joins (and hence κ-meets) and the category of Stone spaces and κ-*continuous* functions.

9.3 Duality for modal μ-algebras

Definition 15. *(Modal μ-homomorphism)* Let M, M' be modal μ-algebras. $h : M \longrightarrow M'$ is a *modal μ-homomorphism* iff it is a homomorphism of modal algebras and

$$h([\![\mu x.t]\!]_A) = [\![\mu x.t]\!]_{h(A)}$$

for all terms t in M and assignments $A : V \longrightarrow M$.

Since we have a representation for modal μ-algebras in terms of modal μ-frames, it follows that the category $\mu\mathsf{Ma}$ of modal μ-algebras and homomorphisms has a dual, which is a category of modal μ-frames and contractions between them, but we do not have an elementary characterization of the maps as yet. The same applies to the case of the category $\mu\mathsf{Ma}^1$ of modal μ-algebras of the first kind. It should be noted that even if the morphisms preserve infinite meets, it does *not* follow that they preserve the (least) fixed points, since the meet of the set of the pre-fixed-points is mapped *into* the set of pre-fixed-points, but it does not follow that the minimum of the image of pre-fixed-points coincides with the minimum of the set of pre-fixed-points in the target algebra. (For example, there can be a pre-fixed-point in the target algebra, which is *below* the homomorphic image of the pre-fixed-points.)

Fortunately, we are in a better position with the second kind of algebras, as can be seen from the following statement. This is at a cost of requiring that modal μ-homomorphisms satisfy the additional condition of being *smooth*.

Definition 16. *(Smooth μ-homomorphism)* Let M, M' be modal μ-algebras of the second kind, and let $h : M \longrightarrow M'$ be a modal μ-homomorphism. h is a *smooth* homorphism iff for every limit ordinal λ, term t and assignment A in M, h preserves the join of all the α unfoldings of the operator $\lambda b.[\![t]\!]_{A(x:=b)}$, for $\alpha < \lambda$. Formally,

$$h(\bigvee_{\alpha<\lambda} (\lambda b.[\![t]\!]_{A(x:=b)})^\alpha(0)) = \bigvee_{\alpha<\lambda} (\lambda b.[\![t]\!]_{hA(x:=b)})^\alpha(0).$$

The intent of the above definition is to prevent the homomorphisms from mapping the interpretation of a fixed point term to the interpretation of a fixed point in the target algebra *without* preserving the joins in the chains of unfoldings approximating the fixed points. (For example, we may have a homomorphism which maps the ω unfolding to the $\omega + 1$ unfolding but nevertheless preserves the least fixed point obtained at some ordinal $\alpha > \omega$.)

We know from Sikorski (see Proposition 14) that a homorphism preserving an infinite join is induced by a continuous function mapping the corresponding defect set to a nowhere dense set under inverse image. This allows us to derive the following characterization of smooth modal μ-homomorphisms.

Proposition 17. *Let M, M' be modal μ-algebras of the second kind, and let $h : M \longrightarrow M'$ be a modal μ-homomorphism. Then h is a smooth μ-homomorphism*

iff its dual h_* maps each of the following defect sets to a nowhere dense set under inverse image.

Defect sets are all

$$\mathrm{Cl}(\bigcup S_{t,\boldsymbol{A},\alpha}) \setminus \bigcup S_{t,\boldsymbol{A},\alpha}$$

where $S_{t,\boldsymbol{A},\alpha} = \{(\lambda b.[\![t]\!]_{A(x:=b)})^\beta \mid \beta < \alpha\}$ for some term t, assignment A in M and ordinal α.

The following duality statement now follows.

Theorem 18. (Duality for modal μ-algebras of the second kind) *The category $\mu\mathsf{Mal}^2$ of the modal μ-algebras of the second kind with smooth homomorphisms is dual to the category $\mu\mathsf{DFra}^2$ of descriptive modal μ-frames of the second kind and contractions mapping subsets of defect sets to nowhere dense sets under inverse image.*

10 Conclusion

We have obtained a completeness result for Kozen's system with respect to a larger class of models, and identified two kinds of modal μ-algebras, possibly corresponding to two different proof systems. We have also shown that one kind of μ-algebras admits a natural duality theorem.

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