Re-interpreting the modal μ-calculus

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Abstract

We re-examine the modal μ-calculus in the light of some classical theory of Boolean algebras and recent results on duality theory for a modal logic with fixed points. We propose interpreting formulas into a field of subsets of states instead of the full power set lattice used by Kozen. Under this interpretation we relate image compact modal frames with Scott continuity of the box modality, m-saturated transition systems and descriptive modal frames. Also, it is shown that the class of image compact modal frames satisfies the Hennessy-Milner property. We conclude by showing that for descriptive modal μ-frames the standard interpretation coincides with the one we proposed.

1 Introduction

Logics can be used to state properties of computer programs. In particular, modal logic finds applications in the specification and verification of programs. Modal logic is essentially a finitary propositional logic with operators $[a]$ expressing after an action a the necessity of truth of a formula. Modal logic enriched with least and greatest fixed points of monotone operators is referred to as the *modal μ-calculus* [Kozen 1983].

Models for the calculus are labeled transition systems; the closed formulas describe sets of states, and the open formulas can be viewed as maps on sets of states. The appealing features of the modal μ-calculus are its decidability (which should be contrasted with the infinitary Hennessy-Milner logic) and its close connection with bisimulation equivalence (under certain restrictions, two states of a transition systems can be shown to be bisimilar if they satisfy the same μ-calculus formulas).

Although known to be decidable — see for example [Kozen 1988] — the modal μ-calculus has not lent itself to an axiomatic characterization. The original calculus introduced in [Kozen 1983] is not yet known to be complete, although the sub-calculus of aconjunctive formulas is shown to be so in [Kozen 1983]. Kozen obtains a completeness result in [Kozen 1988] by introducing an infinitary proof rule. Walukiewicz [Walukiewicz 1993] obtains a similar result with the addition of an induction schema.

Recently, two completeness results for Kozen's original (finitary and infinitary) systems were obtained by one of the authors [Ambler et al. 1994, Ambler et al. 1995] as a corollary of a duality theorem for a certain class of

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modal algebras. The completeness results are for the full calculus, but with respect to larger classes of models (the *modal*, also called *general*, frames, rather than the indexed Kripke frames). The distinguishing feature is that in a modal frame the interpretation is restricted to some algebra of sets of states (also called finitely observable) rather than to the full power set of states. This restriction serves a dual purpose of being a modal algebra of allowable basic propositions, as well as a clopen basis for a topology on the set of states. However, to obtain a full duality statement it was found necessary to restrict the modal frames to those which are descriptive in the sense of [Goldblatt 1976]: the topological space induced by a descriptive modal frame is a Stone space, and the transition relations are in one-to-one correspondence with the derived modalities. For example, the canonical models for Kozen's calculi are descriptive.

In this paper we re-examine some results on the modal μ-calculus in the light of the above-mentioned completeness results. Our starting point is therefore an interpretation of the modal μ-calculus into a field of subsets (the finitely observable propositions) of states rather than into the full power set of states. Using the field of subsets to generate a topology allows us to talk about topological properties of the transition system. For example, we relate m-saturation with image compactness, the latter with Scott continuity of the induced modalities, and point closure with topological continuity. As a consequence of the above relationships we obtain that descriptive modal frames are equivalent to Stone spaces together with a set of image compact transition relations. The latter equivalence is very important because Stone spaces come from compact ultrametric spaces. Therefore, every descriptive modal frame is equivalent to image compact transition relations on a compact ultrametric space, which are widely used in the semantics of concurrent programs.

The paper proceeds as follows. In section 2 we introduce some basic topological notions used in the remainder of the paper. In section 3 we define the language of the modal μ-calculus and two logical systems associated with the language. An interpretation for the language is given in section 4. In section 5 image compact modal frames are studied and the relationship between modal calculus and bisimulation is re-examined and extended to image compact transition systems. After recalling the duality theory for modal μ-algebras in section 6, we relate the two interpretations of the modal μ-calculus.

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2 Mathematical Preliminaries

In this section we provide some basic notions and facts on topological spaces and Boolean algebras. For a more detailed discussion consult [Kuratowski 1966, Sikorski 1969].

A *topology* $\mathcal{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. Every topology $\mathcal{O}(X)$ is ordered by subset inclusion. The pair $(X, \mathcal{O}(X))$ is called a *topological space* and every $o \in \mathcal{O}(X)$ is called an *open set* of the space X. Given an open set $o \in \mathcal{O}(X)$, its complement $c = X \setminus o$ is called a *closed set*. The collection of all closed sets of a topological space $(X, \mathcal{O}(X))$ is denoted by $\mathcal{C}(X)$ and is closed under finite unions and arbitrary intersections. A set $t \subseteq X$ which is both open and closed is said to be *clopen*.

For every set $P \subseteq X$ define its closure $\mathrm{Cl}(P)$ as the intersection of all closed sets $c \in \mathcal{C}(X)$ such that $P \subseteq c$. Dually, the interior of P is the complement of the closure of $X \setminus P$. A set $q \subseteq X$ is *saturated* if q is the intersection of all open sets $o \in \mathcal{O}(X)$ such that $q \subseteq o$. The collection of the saturated sets is denoted by $\mathcal{Q}(X)$. It is a complete lattice since it is closed under arbitrary intersections. A subset $P \subseteq X$ is *compact* in $\mathcal{O}(X)$ if for every directed set $D \subseteq \mathcal{O}(X)$ such that $P \subseteq \bigcup D$ there exists $o \in D$ such that $P \subseteq o$.

A topological space is $\mathcal{T}_0$ if for each pair of distinct points there exists an open set which contains one of the points and does not contain the other; it is $\mathcal{T}_1$ if for each pair of distinct points x, y there exists an open set o such that $x \in o$ and $y \notin o$. A topological space is $\mathcal{T}_2$ or *Hausdorff* if for each pair of distinct points x, y there exist two disjoint open sets containing x, y respectively. A topological space $(X, \mathcal{O}(X))$ is *sober* if there is a bijection between X and sets of the form $\{o \in \mathcal{O}(X) \mid x \in o\}$.

A *base* of a topology $\mathcal{O}(X)$ on X is a set $T \subseteq \mathcal{O}(X)$ such that every open set is the union of elements of T. Often we use $(X, \mathcal{O}(T))$ to denote the topological space $(X, \mathcal{O}(X))$ with base T.

A set $P \subseteq X$ is *connected* if the only clopen sets in the subspace topology are $\emptyset$ and P. A topological space $(X, \mathcal{O}(X))$, is *totally disconnected* if and only if the only connected subsets are the singleton sets. It is *totally separated* if and only if every pair of distinct points $x, y \in X$ can be separated by a clopen subset, that is, there exists a clopen set t such that $x \in t$ and $y \notin t$. A space is a *Stone space* if and only if it is compact, Hausdorff and totally disconnected, or, equivalently, compact and totally separated. Every Stone space is sober.

By a *field of subsets* we shall mean a family of subsets closed under the set-theoretic operations of intersection, union and complement. A field of subsets T is *reduced* if and only if every pair of distinct points is separated by a set $t \in T$, and *perfect* if and only if every ultrafilter of T is of the form $\{t \in T \mid x \in t\}$ for some x, where a subset $U \subseteq B$ of a Boolean algebra B

is an *ultrafilter* if it is upper closed, closed under finite meets and, for every $b \in B$, either $b \in U$ or $\neg b \in U$. A field T of subsets of X is perfect and reduced if and only if the topology on X defined by taking T as a clopen basis is a Stone space.

The *Stone space* $\mathbf{Spec}B$ of a Boolean algebra B is defined by taking the set of all ultrafilters $U(B)$ as points, and the family $\{\beta(b) \mid b \in B\}$ where $\beta(b) = \{U \in U(B) \mid b \in U\}$ as the clopen basis. There exists an isomorphism of B onto the family $\beta(B)$ of the clopen subsets of the Stone space of B. If T is the field of the clopen subsets of a Stone space X then X is the Stone space of T.

3 Modal μ-Calculus

We take $(a \in)\mathrm{Act}$ to be a fixed finite set of *actions* and $(v \in)V$ to be a countable set of *variables*. The set of *formulas* $(\phi \in)\mathcal{F}_\mu$ of the modal μ-calculus are defined by the following syntax:

$$\phi ::= v \mid \underline{0} \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid [a]\phi \mid \mu v.\phi\,,$$

where $a \in \mathrm{Act}$, $v \in V$ and v *is positive in* ϕ in $\mu v.\phi$, that is, all free occurrence of v in ϕ lie within the scope of an even number of negations. This restriction implies monotonicity when interpreting formulas on modal algebras. We denote by $\mathcal{F} \subseteq \mathcal{F}_\mu$ the set of modal formulas, that is, formulas $\phi \in \mathcal{F}_\mu$ which do not contain subformulas of the form $\mu v.\psi$. The connectives $\vee$ and $\langle a\rangle$ are the usual duals to $\wedge$ and $[a]$ respectively. In particular,

$$\langle a\rangle\phi =_{def} \neg[a]\neg\phi.$$

The variable-binding operator μ denotes the least fixed point. The greatest fixed point ν is defined as the dual of the least fixed point as follows:

$$\nu v.\phi =_{def} \neg\mu v'.\neg\phi(v := \neg v'),$$

where $\phi(v := \psi)$ is the formula resulting from replacing all free occurrences of v in ϕ with the formula ψ.

We give a sequent style presentation of the system as in [Kozen 1983]. The structural rules and rules for the basic connectives are those for Gentzen's system **LK** for classical logic [Gentzen 1969]. We augment these with the rules for minimal modal action logic:

$$\frac{\{\psi \mid [a]\psi \in \Gamma\} \vdash \phi, \{\psi' \mid \langle a\rangle\psi' \in \Delta\}}{\Gamma \vdash [a]\phi, \Delta}$$

together with the following rules for the μ-formulas:

$$\frac{\phi(v := \psi) \vdash \psi}{\mu v.\phi \vdash \psi}\ v \text{ not free in } \psi \qquad\qquad \frac{\Gamma \vdash \phi(v := \mu v.\phi), \Delta}{\Gamma \vdash \mu v.\phi, \Delta}$$

The rules (μL) and (μR) can be explained as stating that the term $\mu v.\phi$ is the *least prefixed point*, when viewing the formula ϕ as a (monotone) map on formulas. The rules for $\langle a \rangle$ and ν-formulas are dual. We denote the above system by L_μ.

For every formula $\phi \in \mathcal{F}_\mu$, variable $v \in V$ and natural number i, let ϕ^i be defined inductively by

$$\phi^0 = \underline{0} \quad \text{and} \quad \phi^{i+1} = \phi(v := \phi^i).$$

We denote by L_μ^{inf} the system L_μ, together with the following infinitary rule [Kozen 1988]:

$$\frac{\phi^i \vdash \psi, \text{ all } i \in \omega}{\mu v.\phi \vdash \psi}$$

The above rule states that $\mu v.\phi$ is the *ω-join* of the set of the i-th unfolding of ϕ for $i \in \omega$.

4 Modal Algebras and Fixed-Point Properties

A *modal algebra* is a pair $M = \langle B, \{\tau_a \mid a \in \text{Act}\}\rangle$ where B is a Boolean algebra and $\{\tau_a \mid a \in \text{Act}\}$ is a set of operations which distribute over finite meets (0 is taken to denote the empty meet). If, in addition, B is a complete Boolean algebra (σ-algebra) then M is said to be a complete modal algebra (modal σ-algebra). Informally, we think of elements of B as abstract properties. An *assignment* $A : V \to B$ is a map which assigns to each variable an element of B. It induces an interpretation function for the modal language $[\![\cdot]\!]_A : \mathcal{F} \to B$ as follows:

$$\begin{array}{lcl}
[\![v]\!]_A & = & A(v) \\
[\![\underline{0}]\!]_A & = & 0 \\
[\![\phi_1 \wedge \phi_2]\!]_A & = & [\![\phi_1]\!]_A \wedge [\![\phi_2]\!]_A \\
[\![\neg\phi]\!]_A & = & \neg[\![\phi]\!]_A \\
[\![[a]\phi]\!]_A & = & \tau_a([\![\phi]\!]_A).
\end{array}$$

Since a modal algebra need not be complete, some problems arise when we want to interpret the modal μ-formulas. In particular, the expected interpretation $\mu v.\phi$ as the least fixed point of the (monotone) function $\lambda b.[\![\phi]\!]_{A[b/v]}$ need not be defined. In order to obtain a well defined interpretation function $[\![\cdot]\!]_A : \mathcal{F}_\mu \to B$ we will consider two special classes of modal algebras depending on how the least fixed point of certain monotone functions is calculated.

Let us at first introduce some basic fixed point properties of monotone maps [Lassez et al. 1982]. Let X be a poset and $f : X \to X$ be a monotone map. The least fixed point of f — denoted by $\text{lfp}(f)$ — if it exists is an element $x \in X$ such that $f(x) = x$ and $x \leq y$ for all other $y \in X$ such that $f(y) = y$. We say that f has the *least fixed point property* if the meet of the set of all prefixed points $\bigwedge\{y \in X \mid f(y) \leq y\}$ exists in X.

Lemma 1 *Let X be a poset and $f : X \to X$ be a monotone map with the least fixed point property. Then* $\text{lfp}(f) = \bigwedge\{y \in X \mid f(y) \leq y\}$.

Proof. Let $S = \{y \in X \mid f(y) \leq y\}$. Since f is monotone we have for all $y \in S$ that $f(\bigwedge S) \leq y$ and hence $f(\bigwedge S) \leq \bigwedge S$. But then $f(f(\bigwedge S)) \leq f(\bigwedge S)$ and hence $f(\bigwedge S) \in S$. Therefore $\bigwedge S \leq f(\bigwedge S)$ and hence $f(\bigwedge S) = \bigwedge S$. It is the least fixed point because if $y \in X$ is such that $f(y) = y$ then $y \in S$ which implies $\bigwedge S \leq y$. □

Sometimes the least fixed point of a monotone function $f : X \to X$ can be characterized as the join of finite unfoldings of f, which are defined, for every $i \in \omega$, by

$$f^0 = \lambda x \in X.x \quad \text{and} \quad f^{i+1} = \lambda x \in X.f(f^i(x)).$$

We say that f has the least fixed point property of the *second ω-kind* if X has a bottom element 0 and $\mathrm{lfp}(f) = \bigvee\{f^i(0) \mid i \in \omega\}$

A function which has the least fixed point property need not have the least fixed point property of the second ω-kind (see, for example, [Kozen 1983] in the context of complete modal algebras). However if a function has the least fixed point property of the second ω-kind then it has also the least fixed point property as shown in the following lemma (due to Simon Ambler).

Lemma 2 *Let X be a poset and $f : X \to X$ be a monotone function. If f has the least fixed point property of the second ω-kind then it also has the least fixed point property.*

Proof. We first show by induction that every prefixed point of f is an upper bound of the set $\{f^i(0) \mid i \in \omega\}$. By hypothesis X has a bottom element 0 and hence the unfolding $f^i(0)$ for every $i \in \omega$ exists in X. Let $y \in X$ be a prefixed point, that is, $f(y) \leq y$. Then $f^0(0) = 0 \leq y$. Assume now $f^i(0) \leq y$. By monotonicity of f and because y is a prefixed point we have $f^{i+1}(0) = f(f^i(0)) \leq f(y) \leq y$. Therefore if $f(y) \leq y$ then $\mathrm{lfp}(f) = \bigvee\{f^i(0) \mid i \in \omega\} \leq y$. Hence $\mathrm{lfp}(f) \leq \bigwedge\{y \in X \mid f(y) \leq y\}$. Since $f(\mathrm{lfp}(f)) = \mathrm{lfp}(f)$ we have that $\mathrm{lfp}(f) \in \{y \mid f(y) \leq y\}$. Therefore $\mathrm{lfp}(f) = \bigwedge\{y \mid f(y) \leq y\}$. □

We are now ready to define modal μ-algebras. A modal algebra M is said to be a modal μ-algebra if for every assignment $A : V \to M$, the following interpretation function $[\![\cdot]\!]_A : \mathcal{F}_\mu \to M$ is well defined:

$$\begin{array}{lcl}
[\![v]\!]_A & = & A(v) \\
[\![\underline{0}]\!]_A & = & 0 \\
[\![\phi_1 \wedge \phi_2]\!]_A & = & [\![\phi_1]\!]_A \wedge [\![\phi_2]\!]_A \\
[\![\neg\phi]\!]_A & = & \neg[\![\phi]\!]_A \\
[\![[a]\phi]\!]_A & = & \tau_a([\![\phi]\!]_A) \\
[\![\mu v.\phi]\!]_A & = & \bigwedge\{b \mid [\![\phi]\!]_{A[b/v]} \leq b\}.
\end{array}$$

Notice that by Lemma 1 we have that $[\![\mu v.\phi]\!]_A = \mathrm{lfp}(\lambda b.[\![\phi]\!]_{A[b/v]})$. A modal μ-algebra M is said to be of the second ω-kind if for every assignment

$A : V \to M$ and $\phi \in \mathcal{F}_\mu$

$$\bigwedge\{ b \mid [\![\phi]\!]_{A[b/v]} \leq b \} \quad = \quad \bigvee\{ (\lambda b.[\![\phi]\!]_{A[b/v]})^i(0) \mid i \in \omega \}.$$

By Lemma 1 the above is equivalent to saying

$$\begin{aligned} [\![\mu v.\phi]\!]_A &= \mathrm{lfp}(\lambda b.[\![\phi]\!]_{A[b/v]}) \\ &= \bigwedge\{ b \mid [\![\phi]\!]_{A[b/v]} \leq b \} \\ &= \bigvee\{ (\lambda b.[\![\phi]\!]_{A[b/v]})^i(0) \mid i \in \omega \}. \end{aligned}$$

We observe the following properties of interpretations on modal μ-algebras.

Lemma 3 *Let $\langle B, \{ \tau_a \mid a \in \mathrm{Act} \} \rangle$ be a modal μ-algebra, $\phi \in \mathcal{F}_\mu$ be a modal μ-formula and $A : V \to B$ be an assignment. Then*

1. $[\![\phi]\!]_{A[b/v]} = [\![\phi(v := \psi)]\!]_A$ *for every* $b = [\![\psi]\!]_A$*;*
2. $[\![\mu v.\phi]\!]_A = \bigwedge\{ [\![\psi]\!]_A \mid [\![\phi]\!]_{A[[\![\psi]\!]_A/v]} \leq [\![\psi]\!]_A \}$.

Proof. The first item can be proved by induction on the structure of ϕ, while the second holds because $[\![\mu v.\phi]\!]_A \in \{ [\![\psi]\!]_A \mid [\![\phi]\!]_{A[[\![\psi]\!]_A/v]} \leq [\![\psi]\!]_A \}$ being the least fixed point of $\lambda b.[\![\phi]\!]_{A[b/v]}$. □

The standard interpretation of the modal μ-calculus [Kozen 1983] coincides with the above interpretation whenever we consider modal algebras $M = \langle B, \{ \tau_a \mid a \in \mathrm{Act} \} \rangle$ for which $B = \mathcal{P}(X)$ for some set X. Such an algebra is often referred to as the *full complex algebra*. Clearly, M is a modal μ-algebra and for every assignment $A : V \to B$

$$[\![\mu v.\phi]\!]_A \quad = \quad \bigcap\{ P \subseteq X \mid [\![\phi]\!]_{A[P/v]} \subseteq P \}.$$

In general, modal μ-algebras arise as Lindenbaum algebras of Kozen's systems L_μ and L_μ^{inf} in the following sense [Ambler et al. 1995].

Theorem 4 *The Lindenbaum algebra of the logical system L_μ is a countable modal μ-algebra while the Lindenbaum algebra of the logical system L_μ^{inf} is a countable modal μ-algebra of the second ω-kind.*

5 Image Compact Modal Frames

In this section we introduce labeled transition systems as models for the modal calculus. By enriching the set of states with a topology we obtain the modal frames. We then study the class of image compact modal frames with respect to Scott continuity of modality, m-saturated transition systems and descriptive modal frames. Finally, we show that image compact modal frames satisfy the Hennessy-Milner property.

A (labeled) *transition system* (also called indexed Kripke frame) is a pair $X = \langle X, \{ \xrightarrow{a} \mid a \in \mathrm{Act} \} \rangle$ where X is a set (of states or configurations), Act an index set (of labels) and for each $a \in \mathrm{Act}$, $\xrightarrow{a} \subseteq X \times X$ is a transition relation. Each transition relation $\xrightarrow{a}$ induces a function (often called box modality) $\tau_a : \mathcal{P}(X) \to \mathcal{P}(X)$ defined, for all $P \subseteq X$, as follows

$$\tau_a(P) \quad = \quad \{ x \in X \mid \forall y. x \xrightarrow{a} y \Rightarrow y \in P \}.$$

Since the pair $\langle \mathcal{P}(X), \{\tau_a \mid a \in \mathrm{Act}\,\}\rangle$ is a modal algebra, for every assignment $A : V \to \mathcal{P}(X)$, a labeled transition system $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}\rangle$ induces an interpretation function $[\![\cdot]\!]_A : \mathcal{F} \to \mathcal{P}(X)$ defined as in the previous section. Therefore, transition systems are models of the modal calculus where propositions are modeled by sets of states. However, this model can be unnecessarily large: if the set of states X is infinite then $\mathcal{P}(X)$ has uncountably many elements, whereas the modal calculus may only denote at most countably many of those, that is, only at most countably many propositions have a syntactic denotation as a formula of $\mathcal{F}$. For a transition system X and assignment $A : V \to \mathcal{P}(X)$ let $T_A(X)$ denote the set of propositions which have a syntactic denotation as modal formulas, that is

$$T_A(X) \quad = \quad \{[\![\phi]\!]_A \mid \phi \in \mathcal{F}\,\}.$$

Clearly, $T_A(X)$ is a field of subsets of X and it is closed under τ_a for every $a \in \mathrm{Act}$. Indeed, $\tau_a([\![\phi]\!]_A) = [\![[a]\phi]\!]_A$ for every $\phi \in \mathcal{F}$. This leads us to the following definition of a modal frame (also called general frame [Hughes and Cresswell 1984]).

Definition 1 Let $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}\rangle$ be a transition system and T be a field of subsets of X. The triple $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}, T\rangle$ is a *modal frame* if $\tau_a(t) \in T$ for each $t \in T$ and $a \in \mathrm{Act}$.

Modal frames are models of the modal calculus in the usual way if we consider assignments $A : V \to T$. Intuitively, propositions in T are those which are finitely observable [Smyth 1983, Abramsky 1991]. Every modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}, T\rangle$ induces a transition system by forgetting the field of sets T and conversely, for every assignment $A : V \to \mathcal{P}(X)$, every transition system $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}\rangle$ induces a modal frame by taking $T_A(X)$ as the field of sets.

A modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}, T\rangle$ is a modal μ-frame (modal μ-frame of the second ω-kind) if the pair $\langle T, \{\tau_a \mid a \in \mathrm{Act}\,\}\rangle$ is a modal μ-algebra (modal μ-algebra of the second ω-kind, respectively).

Moving from transition systems $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}\rangle$ to modal frames $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\,\}, T\rangle$ allows us to consider topological properties of the transition system because the field of subsets T can be used as a clopen basis for the topological space $(X, \mathcal{O}(T))$. In the sequel we will consider some of these properties and we will relate them to properties of the transition system. Of special interest is the case when the topological space $(X, \mathcal{O}(T))$ is a Stone space because of the strong relationships between them and compact ultrametic spaces (we refer the interested readers to [Smyth 1992]). Labeled transition systems with compact ultrametric spaces on the set of states are used in the semantics of concurrent languages (see [Bakker and Rutten 1992] for an overview and [Breugel 1994] for image compact transition systems).

We start with the relationships between image compact transition systems and Scott continuity of the box operators. Recall that a transition system $X = \langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}\rangle$ is *image finite* if the set

$$Im(x,a) \quad = \quad \{y \in X \mid x \xrightarrow{a} y\ \}$$

is finite for all $x \in X$ and $a \in \mathrm{Act}$. If the set X comes equipped with a topology $\mathcal{O}(X)$, we say that the transition system X is *image compact* if every set $Im(x,a)$ is compact in the topology $\mathcal{O}(X)$. Clearly, every image finite transition system is image compact but the converse does not hold in general. In the sequel we will call a modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, T\rangle$ image compact if the underlying transition system is image compact in the induced topology $\mathcal{O}(T)$.

In [Stirling 1992, Lemma 2.5.1] it is shown that a transition system is image finite if and only if the induced functions $\tau_a : \mathcal{P}(X) \to \mathcal{P}(X)$ are Scott continuous. This results can be generalized to image compact transition system as follows.

Lemma 5 *Let $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, T\rangle$ be a modal frame. The transition relation $\xrightarrow{a}$ is image compact in $(X, \mathcal{O}(T))$ if and only if $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous.*

Proof. Let $D \subseteq \mathcal{O}(T)$ be a directed set. We have:

$$\begin{aligned} \tau_a(\textstyle\bigcup D) &= \{x \in X \mid Im(x,a) \subseteq \textstyle\bigcup D\ \} \\ &= \textstyle\bigcup\{\{x \in X \mid Im(x,a) \subseteq o\ \} \mid o \in D\ \} && [Im(x,a) \text{ is compact}] \\ &= \textstyle\bigcup\{\tau_a(o) \mid o \in D\ \}. \end{aligned}$$

Therefore τ_a is Scott continuous. It remains to prove it is well defined, that is $\tau_a(o) \in \mathcal{O}(T)$ for all $o \in \mathcal{O}(T)$. Let $o \in \mathcal{O}(T)$. Then $o = \bigcup\{t \in T \mid t \subseteq o\ \}$ because T forms a basis for the topology $\mathcal{O}(T)$. Clearly, the set $\{t \in T \mid t \subseteq o\ \}$ is directed, hence we have

$$\begin{aligned} \tau_a(o) &= \tau_a(\textstyle\bigcup\{t \in T \mid t \subseteq o\ \}) \\ &= \textstyle\bigcup\{\tau_a(t) \mid t \subseteq o \;\&\; t \in T\ \}. \end{aligned}$$

Since $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, T\rangle$ is a modal frame, we have by definition that $\tau_a(t) \in T$ for all $t \in T$, therefore $\bigcup\{\tau_a(t) \mid t \subseteq o \;\&\; t \in T\ \} \in \mathcal{O}(T)$.

Conversely, take $x \in X$ and let $D \subseteq \mathcal{O}(T)$ be a directed set.

$$\begin{aligned} & Im(x,a) \subseteq \textstyle\bigcup D \iff x \in \tau_a(\textstyle\bigcup D) && [\text{definition } \tau_a] \\ \iff\ & x \in \textstyle\bigcup\{\tau_a(o) \mid o \in D\ \} && [\tau_a \text{ Scott continuous}] \\ \iff\ & \exists o \in D. x \in \tau_a(o) \\ \iff\ & \exists o \in D. Im(x,a) \subseteq o, \end{aligned}$$

that is, $Im(x,a)$ is compact in $\mathcal{O}(T)$. □

The direction from left to right of the above lemma is known — for the restricted class of descriptive (or perfect) modal frames — as the (dual of) Esakia's Lemma [Esakia 1974] (see also [Sambin and Vaccaro 1989]).

In the presence of a topology, it is often important to consider continuous functions. If we work with relations, there are at least three kinds of continuity depending on whether we consider open or closed sets. Recall [Michael 1951,Berge 1963] that for a given topological space $(X, \mathcal{O}(X))$, a relation $\xrightarrow{a} \subseteq X \times X$ is

> *upper semi-continuous* if $\tau_a(o) \in \mathcal{O}(X)$ for every open set $o \in \mathcal{O}(X)$;
> *lower semi-continuous* if $\tau_a(c) \in \mathcal{C}(X)$ for every closed set $c \in \mathcal{C}(X)$;
> *continuous* if it is both upper and lower semi-continuous.

Every relation $\xrightarrow{a} \subseteq X \times X$ is clearly equivalent to the function $Im(\cdot, a) : X \to \mathcal{P}(X)$. Moreover, the three notions of continuity of the relation $\xrightarrow{a}$ correspond to the standard notions of continuity of the function $Im(\cdot, a)$ if we topologize $\mathcal{P}(X)$ with the upper, lower or Vietoris topology respectively (for more information we refer the interested reader to [Michael 1951],[Smyth 1983] and [Bonsangue and Kok 1994]). The above (standard) definition of continuity of a relation $\xrightarrow{a}$ does *not* coincide with the notion of continuity given in [Sambin and Vaccaro 1988]: for a given field T of subsetes of X, a relation $\xrightarrow{a} \subseteq X \times X$ is continuous in $(X, \mathcal{O}(T))$ if $\tau_a(t) \in T$ for every $t \in T$, that is, the triple $\langle X, \xrightarrow{a}, T \rangle$ is a modal frame. The latter notion of continuity implies lower semi-continuity but it is, in general, incomparable with upper semi-continuity [Sambin and Vaccaro 1988]. However, in image compact modal frames the transition relations are continuous.

Lemma 6 *If $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T \rangle$ is an image compact modal frame then $\xrightarrow{a}$ is continuous and $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous.*

Proof. By Lemma 5, the relation $\xrightarrow{a}$ is upper semi-continuous and $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous. Therefore we only need to prove that $\tau_a(c) \in \mathcal{C}(T)$ for every closed set $c \in \mathcal{C}(T)$. Let $c \in \mathcal{C}(T)$ and say $o = X \setminus c \in \mathcal{O}(T)$. Because T is a basis for $\mathcal{O}(T)$ there exists $S \subseteq T$ such that $o = \bigcup S$. Hence $c = \bigcap \{X \setminus t \mid t \in S\}$. It is now straightforward to verify that, by definition, τ_a preserves arbitrary intersections and hence $\tau_a(c) = \bigcap \{\tau_a(X \setminus t) \mid t \in S\}$. Since T is a field of subsets of X we have that $X \setminus t \in T$ for each $t \in S$ and this, together with the fact that F is a modal frame, ensures $\tau_a(X \setminus t) \in T$ for every $t \in S$. Because T is closed under complement we have that it is a clopen basis for $\mathcal{O}(T)$. Therefore, $\tau_a(X \setminus t)$ is closed and so is $\tau_a(c) = \bigcap \{\tau_a(X \setminus t) \mid t \in S\}$ because arbitrary intersections of closed sets are closed. $\square$

The implication of the above lemma becomes an equivalence if T is a reduced and perfect field of subsets of X. If this is the case then $(X, \mathcal{O}(T))$ is a Stone space, and hence compact and Hausdorff. Recall that for such spaces a set is compact if and only if it is closed. Also, if X is a Stone space then T coincides with the set of all compact opens, or equivalently, with the set of all clopens of $\mathcal{O}(T)$.

Lemma 7 *Let T be a reduced and perfect field of subsets of X and $\xrightarrow{a}$ be* *binary relations on X for all $a \in$ Act. The triple $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}, T\rangle$ is an image compact modal frame if and only if for all $a \in$ Act, $\xrightarrow{a}$ is continuous and $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous.*

Proof. If $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}, T\rangle$ is a modal frame with $\xrightarrow{a}$ image compact then, by Lemma 6, $\xrightarrow{a}$ is continuous and $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous.

Conversely, assume $\xrightarrow{a}$ continuous. To prove $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}, T\rangle$ is a modal frame we need to show $\tau_a(t) \in T$ for all $t \in T$, or, equivalently, $\tau_a(t)$ is both open and closed. But this is immediate because $\xrightarrow{a}$ is continuous and $t \in T$ is clopen. Also, since $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous then, by Lemma 5, $\xrightarrow{a}$ is image compact. $\square$

Image compact modal frames are very close to m-saturated transition systems. Recall that for a given assignment $A : V \to \mathcal{P}(X)$, a transition system $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}\rangle$ is called *m-saturated* if for every set of modal formulas $\Phi \subseteq \mathcal{F}$, $a \in$ Act and $x \in X$ such that for all finite $\Psi \subseteq \Phi$ there exists $y \in Im(x, a)$ with $y \in \bigcap\{[\![\psi]\!]_A \mid \psi \in \Psi\}$, then there exists $y \in Im(x, a)$ with $y \in \bigcap\{[\![\phi]\!]_A \mid \phi \in \Phi\}$ (see [Visser 1994] and [Fine 1973] where this notion originally stems from).

Lemma 8 *Let $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}, T\rangle$ be an image compact modal frame. For all assignments $A : V \to T$ the transition system $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}\rangle$ is m-saturated.*

Proof. Let $A : V \to T$ and suppose $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}\rangle$ is not m-saturated. Then there exists a set of modal formulas $\Phi \subseteq \mathcal{F}$ such that for all finite $\Psi \subseteq \Phi$, $Im(x, a) \cap \bigcap\{[\![\psi]\!]_A \mid \psi \in \Psi\} \neq \emptyset$ but $Im(x, a) \cap \bigcap\{[\![\phi]\!]_A \mid \phi \in \Phi\} = \emptyset$. Hence $Im(x, a) \subseteq \bigcup\{[\![\neg\phi]\!]_A \mid \phi \in \Phi\}$. But $Im(x, a)$ is compact and therefore there exists a finite $\Psi \subseteq \Phi$ such that $Im(x, a) \subseteq \bigcup\{[\![\neg\psi]\!]_A \mid \psi \in \Psi\}$, or, equivalently, $Im(x, a) \cap \bigcap\{[\![\psi]\!]_A \mid \psi \in \Psi\} = \emptyset$. Contradiction! $\square$

At the beginning of this section we have seen that for a fixed assignment, every transition system induces a modal frame. By using this construction we can prove the converse of the previous lemma.

Lemma 9 *Let $A : V \to \mathcal{P}(X)$ be an assignment and $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}\rangle$ be an m-saturated transition system. Then $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}, T_A(X)\rangle$ is an image compact modal frame.*

Proof. Take $P \subseteq T_A(X)$ such that $Im(x, a) \subseteq \bigcup P$. Then by definition of $T_A(X)$ there exists $\Phi \subseteq \mathcal{F}$ such that $P = \{[\![\phi]\!]_A \mid \phi \in \Phi\}$. Suppose now $Im(x, a)$ is not compact. Then for all finite $\Psi \subseteq \Phi$, $Im(x, a) \not\subseteq \bigcup\{[\![\psi]\!]_A \mid \psi \in \Psi\}$ and hence $Im(x, a) \cap \bigcap\{[\![\neg\psi]\!]_A \mid \psi \in \Psi\} \neq \emptyset$. But $\langle X, \{\xrightarrow{a} \mid a \in \text{Act}\}\rangle$ is m-saturated, and hence there is $y \in Im(x, a)$ such

that $y \in \bigcap\{[\![\neg\phi]\!]_A \mid \phi \in \Phi\}$ contradicting $Im(x,a) \subseteq \bigcup\{[\![\phi]\!]_A \mid \phi \in \Phi\}$.
□

Next we continue with the relationship between the continuity of the transition relations and a property of the transition system which is often referred to as *point closure* [Sambin and Vaccaro 1988]. A modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ is said to be point closed if the set $Im(x,a)$ is closed in the topology $\mathcal{O}(T)$ for all $x \in X$ and $a \in \mathrm{Act}$.

Lemma 10 *If $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ is a modal frame then for every $x \in X$ and $a \in \mathrm{Act}$ the following are equivalent:*

1. *$Im(x,a)$ is closed in the topology $\mathcal{O}(T)$;*
2. *$x \xrightarrow{a} y \iff \forall t \in T. x \in \tau_a(t) \Rightarrow y \in t$.*

Proof. See Proposition (8), page 267 in [Sambin and Vaccaro 1988]. □

If the field T is reduced then $(X, \mathcal{O}(T))$ is a Hausdorff space, and hence every compact set is closed. Therefore by the above lemma we have that if $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ is an image compact modal frame and T is a reduced field of subsets, then

$$x \xrightarrow{a} y \iff \forall t \in T. x \in \tau_a(t) \Rightarrow y \in t.$$

for all $x \in X$ and $a \in \mathrm{Act}$. Modal frames of this kind are often called *refined frames* [Thomason 1972].

In order to prove the converse of Lemma 10 we need at first the following lemma, which gives a way to extend, for a fixed field T of subsets of X, a finite meet preserving function $f : T \to T$ to an arbitrary meet preserving function $f^E : \mathcal{P}(X) \to \mathcal{P}(X)$ that is Scott continuous on open sets and such that $f^E(t) = f(t)$ for every $t \in T$. A canonical extension for Boolean algebras is known from [Jonsson and Tarski 1951]. Recently, a similar construction has been given for frames (the pointless counterpart of sober topological spaces) in [Bonsangue et al. 1995]. Below we give a slight modification of the Jonsson and Tarski's extension theorem.

Lemma 11 *Let T be a perfect and reduced field of subsets of X and $f : T \to T$ be a function preserving finite intersections. Define the functions $\hat{f} : \mathcal{O}(T) \to \mathcal{O}(T)$ and $f^E : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows*

$$\hat{f}(o) = \bigcup\{f(t) \mid t \in T \;\&\; t \subseteq o\} \text{ for all } o \in \mathcal{O}(T), \text{ and}$$
$$f^E(P) = \bigcap\{\hat{f}(o) \mid o \in \mathcal{O}(T) \;\&\; P \subseteq o\} \text{ for all } P \in \mathcal{P}(X).$$

Then the function $\hat{f}$ is well defined, Scott continuous, preserves finite intersections and $\hat{f}(t) = f(t)$ for all $t \in T$, while the function f^E preserves arbitrary intersections and $f^E(o) = \hat{f}(o)$ for all $o \in \mathcal{O}(T)$.

Proof. For every open set $o \in \mathcal{O}(T)$ the function $\hat{f}(o) \in \mathcal{O}(T)$ because it is the union of sets of the form $f(t) \in T$ which are open. Preservation of finite intersections is easily verified and clearly $\hat{f}(t) = f(t)$ for all $t \in T$.

It remains to prove it is Scott continuous. Let $D \subseteq \mathcal{O}(T)$ be a directed set and recall that since $(X, \mathcal{O}(T))$ is a Stone space then every $t \in T$ is compact open. Then we have

$$\begin{aligned} \hat{f}(\bigcup D) &= \bigcup\{ f(t) \mid t \subseteq \bigcup D \} \\ &= \bigcup\{ \bigcup_{o \in D} f(t) \mid t \subseteq o \} && [t \text{ is compact}] \\ &= \bigcup_{o \in D} \bigcup\{ f(t) \mid t \subseteq o \} \\ &= \bigcup_{o \in D} \hat{f}(o). \end{aligned}$$

The proof that f^E preserves arbitrary intersections is essentially (the dual of) Theorem 2.4 in [Jonsson and Tarski 1951]. An alternative proof can be found in [Bonsangue et al. 1995] (Lemma 3.9) because X is a sober space (being a Stone space) and $\mathcal{P}(X) = \mathcal{Q}(X)$ because X is $\mathcal{T}_1$ (again X being a Stone space). □

We are now ready to prove the converse of Lemma 10 under the restriction of allowing only perfect and reduced field of subsets.

Lemma 12 *Let T be a perfect and reduced field of subsets of X and $\{ f_a : T \to T \mid a \in \mathrm{Act} \}$ be a set of functions on T preserving finite intersections. For every $a \in \mathrm{Act}$ define the transition relations $\xrightarrow{a} \subseteq X \times X$ by*

$$\forall x, y \in X . x \xrightarrow{a} y \iff (\forall t \in T . x \in \tau_a(t) \Rightarrow y \in t).$$

Then $\langle X, \{ \xrightarrow{a} \mid a \in \mathrm{Act} \}, T \rangle$ is an image compact modal frame and $\tau_a = f_a^E$ for every $a \in \mathrm{Act}$.

Proof. It is enough to prove $\tau_a = f_a^E$ because, by Lemma 11, $\tau_a(t) = f_a^E(t) = f_a(t) \in T$ for every $t \in T$ implies $\langle X, \{ \xrightarrow{a} \mid a \in \mathrm{Act} \}, T \rangle$ is a modal frame. Also, $\tau_a(o) = f_a^E(o) = \hat{f}_a(o)$ implies $\tau_a : \mathcal{O}(T) \to \mathcal{O}(T)$ is Scott continuous and hence, by Lemma 5, the relations $\xrightarrow{a}$ are image compact.

Suppose $\bigcap\{ t \in T \mid x \in f_a(t) \} = \bigcap\{ Q \subseteq X \mid x \in f_a^E(Q) \}$ for all $x \in X$. Then we would have, for every $P \subseteq X$,

$$\begin{aligned} \tau_a(P) &= \{ x \in X \mid \forall y . x \xrightarrow{a} y \Rightarrow y \in P \} \\ &= \{ x \in X \mid \bigcap\{ t \in T \mid x \in f_a(t) \} \subseteq P \} \\ &= \{ x \in X \mid \bigcap\{ Q \mid x \in f_a^E(Q) \} \subseteq P \} \\ &\stackrel{*}{=} \{ x \in X \mid x \in f_a^E(P) \} \\ &= f_a^E(P). \end{aligned}$$

where $\stackrel{*}{=}$ holds because f_a^E preserves arbitrary intersections (see Lemma 5.4 in [Bonsangue and Kok 1994]). Therefore it is enough to prove $\bigcap\{ t \in T \mid x \in f_a(t) \} = \bigcap\{ Q \subseteq X \mid x \in f_a^E(Q) \}$ for all $x \in X$.

($\subseteq$) Suppose for all $t \in T$ if $x \in f_a(t)$ then $y \in t$ and let $x \in f_a^E(Q) = \bigcap\{ \hat{f}_a(o) \mid Q \subseteq o \}$ for some $Q \subseteq X$. Therefore, $x \in \hat{f}_a(o)$ for all $Q \subseteq o$. But $\hat{f}_a(o) = \bigcup\{ f_a(t) \mid t \subseteq o \}$, and hence there exists $t \subseteq o$ such that $x \in f_a(t)$. By hypothesis this implies $y \in t \subseteq o$ and therefore $y \in Q = \bigcap\{ o \mid Q \subseteq o \}$.

($\supseteq$) Suppose for all $Q \subseteq X$ if $x \in f_a^E(Q)$ then $y \in Q$. It follows that for all $t \in T$ if $x \in f_a(t) = f_a^E(t)$ then $y \in t$.

$\square$

We conclude this series of relationships by considering descriptive modal frames [Goldblatt 1976]. A modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ is said to be *descriptive* if T is perfect and reduced, and for all $a \in \mathrm{Act}$

$$\forall x, y \in X. x \xrightarrow{a} y \iff (\forall t \in T. x \in \tau_a(t) \Rightarrow y \in t).$$

Theorem 13 *Let T be a perfect and reduced field of subsets of X and $\{\xrightarrow{a} \mid a \in \mathrm{Act}\}$ be a set of transition relations on X. The following are equivalent:*

1. $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ *is a descriptive modal frame;*
2. $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ *is an image compact modal frame;*
3. *for all $a \in \mathrm{Act}$ the induced modalities τ_a preserve opens, closed sets and directed families of opens (with respect to the topology $\mathcal{O}(T)$).*

Proof. The equivalence between 1 and 2 is a direct consequence of Lemma 10 and Lemma 12. The equivalence between 2 and 3 follows from Lemma 7. $\square$

Every image compact modal frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ satisfies the Hennessy-Milner property, that is, two states satisfy the same modal formulas if and only if they are bisimilar in $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}\rangle$. Recall that for an assignment $A : V \to T$ and for a transition system $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}\rangle$, two states $x, y \in X$ are bisimilar if there exists a bisimulation relation $R \subseteq X \times X$ with $(x, y) \in R$, where R is a bisimulation relation if $(x, y) \in R$ implies for all actions $a \in \mathrm{Act}$ and variables $v \in V$,

$$\begin{array}{ll} (a) & x \in A(v) \iff y \in A(v), \\ (b) & \forall x'. x \xrightarrow{a} x' \Rightarrow \exists y'. y \xrightarrow{a} y' \ \&\ (x', y') \in R, \\ (c) & \forall y'. y \xrightarrow{a} y' \Rightarrow \exists x'. x \xrightarrow{a} x' \ \&\ (x', y') \in R. \end{array}$$

Lemma 14 *Let $\langle X, \{\xrightarrow{a}\}, T\rangle$ be an image compact modal frame and let $A : V \to T$ be an assignment. Two states $x, y \in X$ are bisimilar if and only if for all modal formulas $\phi \in \mathcal{F}$*

$$x \in [\![\phi]\!]_A \iff y \in [\![\phi]\!]_A$$

Proof. Follows easily from the Lemmas 8, 9 and [Visser 1994]. $\square$

For a complete overview of properties of transition systems in connection with the Hennessy-Milner property we refer the interested reader to [Hollenberg 1995]. Since $\mathcal{F} \subseteq \mathcal{F}_\mu$ we have as corollary the following result for the modal μ-calculus.

Corollary 15 *Let $\langle X, \{\xrightarrow{a}\}, T\rangle$ be an image compact modal μ-frame and let $A : V \to T$ be an assignment. For all $x, y \in X$ if*

$$\forall \phi \in \mathcal{F}_\mu . x \in [\![\phi]\!]_A \iff y \in [\![\phi]\!]_A$$

then x, y are bisimilar.

It is an open problem if the converse of the above corollary holds for general modal frames. If we use the standard interpretation function, or, equivalently, if we consider only modal frames of the form $\langle X, \{\xrightarrow{a}\}, \mathcal{P}(X)\rangle$ then two states are bisimilar implies they satisfy the same modal μ-formulas [Stirling 1992].

6 Duality Theory for the Modal μ-Calculus

Duality theory originates from Stone, who set out to formulate a representation theorem for Boolean algebras as a field of sets so that the operations of conjunction, disjunction and negation can be modeled by their set-theoretic counterparts. Stone proposed a construction of a topological space $\mathbf{Spec}B$ from an arbitrary Boolean algebra B taking as points the ultrafilters of B — denoted $\mathcal{U}(B)$ — and taking as basis of the topology the collection of sets

$$\mathcal{U}_b \quad = \quad \{\mathcal{U} \in \mathcal{U}(B) \mid b \in \mathcal{U}\}$$

for every $b \in B$. This gives rise to order-isomorphisms:

$$\begin{array}{lcll} B & \cong & \mathrm{Clop}\mathbf{Spec}B & \beta : b \mapsto \{\mathcal{U} \in \mathcal{U}(B) \mid b \in \mathcal{U}\}; \\ X & \cong & \mathbf{Spec}\mathrm{Clop}X & \gamma : s \mapsto \{t \in \mathrm{Clop}X \mid s \in t\}, \end{array}$$

where X is a Stone space. It was observed later that this correspondence can be expressed as an equivalence of categories ($\mathbf{Bool}^{op} \simeq \mathbf{Stone}$); for more detail see [Johnstone 1982].

This line of research has been followed by one of the authors in [Ambler et al. 1994, Ambler et al. 1995] to obtain a duality theorem for modal μ-algebras. Of course, the fact that Boolean algebras are endowed with additional structure — in terms of an indexed family of modalities — must be considered. To cater for this, we use the framework of modal duality as originally proposed in [Halmos 1962] (later developed in [Sambin and Vaccaro 1988]) and extend this to give an account of the fixed point operators. The origins of this framework go back to the Jonsson and Tarski representation for Boolean algebras with operators [Jonsson and Tarski 1951].

Given a modal frame $F = \langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ we have already seen in the previous section that the pair $F_* = \langle T, \{\tau_a \mid a \in \mathrm{Act}\}\rangle$ is a modal algebra. Conversely, if $M = \langle B, \{\tau_a \mid a \in \mathrm{Act}\}\rangle$ is a modal algebra then $M^* = \langle \mathcal{U}(B), \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, \mathrm{Clop}\mathbf{Spec}B\rangle$ is a descriptive modal frame, where for every $a \in \mathrm{Act}$ the transition relations $\xrightarrow{a}$ are defined by

$$\forall \mathcal{U}_1, \mathcal{U}_1 \in \mathcal{U}(B) \,.\, \mathcal{U}_1 \xrightarrow{a} \mathcal{U}_2 \quad \Longleftrightarrow \quad (\forall b \in B . \tau_a(b) \in \mathcal{U}_1 \Rightarrow b \in \mathcal{U}_2).$$

Moreover, the two construction are each other's inverse, that is, F is isomorphic to $(F_*)^*$ while M is isomorphic to $(M^*)_*$ (for more we refer the reader to [Sambin and Vaccaro 1988]).

Observe now that for a given Boolean algebra B, the isomorphism $\beta : B \to \mathrm{Clop}\mathbf{Spec}B$ is order preserving. Therefore, if M is a modal μ-algebra then M^* is a descriptive modal μ-frame. Similarly, if M is a modal μ-algebra of the second ω-kind, then M_* is a descriptive modal μ-frame of the second ω-kind.

In [Ambler et al. 1995] it is shown that the above representation of modal μ-algebras can be extended to a full duality statement by considering opportune sub-categories of those for modal algebras and modal frames considered in [Halmos 1962] and [Sambin and Vaccaro 1988]. Also, the representation of modal μ-algebras can be applied to the Lindenbaum algebras of Kozen's systems L_μ and L_μ^{inf} giving rise to modal μ-frames which serve as *canonical* models. Modal μ-frames of Lindenbaum algebras of the systems L_μ and L_μ^{inf} are also used to prove the following completeness results [Ambler et al. 1994, Ambler et al. 1995].

Theorem 16 *The logical system L_μ is complete with respect to the class of modal μ-frames, while the logical system L_μ^{inf} is complete with respect to modal μ-frames of the second-ω-kind.*

Next we show that for descriptive modal μ-frames, the standard interpretation of the modal μ-calculus [Kozen 1983] coincides with the interpretation given in Section 4. We need at first the following lemma.

Lemma 17 *Let $F = \langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\}, T\rangle$ be a descriptive modal μ-frame. Then*

$$[\![\mu v.\phi]\!]_A = \bigcap\{[\![\psi]\!]_A \mid [\![\phi]\!]_{A[[\![\psi]\!]_A/v]} \subseteq [\![\psi]\!]_A\}.$$

for every assignment $A : V \to T$, $v \in V$ and $\phi \in \mathcal{F}_\mu$.

Proof. Since the field T is perfect and reduced then $(X, \mathcal{O}(T))$ is a Stone space. Therefore we have the following isomorphism

$$\begin{array}{lcll} T & \cong & \mathrm{Clop}\mathbf{Spec}T & \beta : t \mapsto \{\mathcal{U} \in \mathcal{U}(T) \mid t \in \mathcal{U}\}, \\ X & \cong & \mathcal{U}(T) & \gamma : x \mapsto \{t \in T \mid x \in t\}. \end{array}$$

Observe that $x \in t$ if and only if $\gamma(x) \in \beta(t)$ if and only if $t \in \gamma(x)$. Let $f = \lambda t.[\![\phi]\!]_{A[t/v]}$, we need to prove

$$[\![\mu v.\phi]\!]_A = \bigcap\{[\![\psi]\!]_A \mid f([\![\psi]\!]_A) \subseteq [\![\psi]\!]_A\}.$$

Since β is an isomorphism and $[\![\mu v.\phi]\!]_A = \bigwedge\{[\![\psi]\!]_A \mid f([\![\psi]\!]_A) \subseteq [\![\psi]\!]_A\}$ because of Lemma 3, the above is equivalent to proving

$$\beta(\bigwedge\{[\![\psi]\!]_A \mid f([\![\psi]\!]_A) \subseteq [\![\psi]\!]_A\}) = \bigcap\{\beta([\![\psi]\!]_A) \mid f([\![\psi]\!]_A) \subseteq [\![\psi]\!]_A\}.$$

The inclusion from left to right follows from monotonicity of β. To establish the reverse inclusion consider an ultrafilter $\mathcal{U}$ belonging to the intersection. Then for all $t = [\![\psi]\!]_A \in T$ such that $f(t) \leq t$ we have $\mathcal{U} \in \beta(t)$, or equivalently $t \in \mathcal{U}$. But by Lemma 3,

$$\mathrm{lfp}(f) = [\![\mu v.\phi]\!]_A = \bigwedge\{[\![\psi]\!]_A \mid f([\![\psi]\!]_A) \leq [\![\psi]\!]_A\},$$

therefore $\mathrm{lfp}(f) \in \mathcal{U}$ and hence $\mathcal{U} \in \beta(\mathrm{lfp}(f))$. $\square$

Similarly, it is straightforward to prove for a descriptive modal μ-frame $F = \langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, T\rangle$ that $[\![\mu v.\phi]\!]_A = \bigcap\{t \in T \mid [\![\phi]\!]_{A[t/v]} \subseteq t\ \}$ for every assignment $A : V \to T$, $v \in V$ and $\phi \in \mathcal{F}_\mu$. However, if F is of the second ω-kind we do not have, in general, that the join of unfoldings coincides with their union.

Theorem 18 *Let $F = \langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, T\rangle$ be a descriptive modal μ-frame. Then*

$$[\![\mu v.\phi]\!]_A = \bigcap\{P \subseteq X \mid [\![\phi]\!]_{A[P/v]} \subseteq P\ \}.$$

for every assignment $A : V \to T$, $v \in V$ and $\phi \in \mathcal{F}_\mu$.

Proof. Consider the modal μ-frame $\langle X, \{\xrightarrow{a} \mid a \in \mathrm{Act}\ \}, \mathcal{P}(X)\rangle$. For for every assignment $A : V \to T \subseteq \mathcal{P}(X)$, we then have by Lemma 3 that $[\![\mu v.\phi]\!]_A = \bigcap\{[\![\psi]\!]_A \mid [\![\phi]\!]_{A[[\![\psi]\!]_A/v]} \subseteq [\![\psi]\!]_A\ \}$. But this is the interpretation of $\mu v.\phi$ in the modal μ-frame F by Lemma 17. $\square$

7 Conclusion

We have introduced image compact modal frames as models for the (finitary) modal calculus. Under certain restrictions they may also serve as models for the modal μ-calculus even if the interpretation function does not coincide with the (standard) interpretation of [Kozen 1983]: the interpretation of infinite disjunction, even in the descriptive modal μ-frames, is not the union, but its closure [Ambler et al. 1995]. Such interpretation of disjunction can be considered unsatisfactory for reasoning about fairness and liveness properties of programs, but this is the price one has to pay for compactness or, equivalently, Scott continuity of modalities. It is possible that an alternative, complete μ-calculus can be found using a topological duality which differs from the Stone duality (such as, for example, the duality proposed in [Bonsangue et al. 1995]).

The two interpretations of modal μ-formulas coincide for descriptive modal μ-frames. Descriptive modal μ-frames are image compact modal μ-frames for which the field of finitely observable properties is perfect and reduced. Kozen's logical systems [Kozen 1983] are complete with respect to certain classes of descriptive modal μ-frames [Ambler et al. 1995].

We have also given some topological properties of image compact modal frames and shown their relationships with model-theoretic properties such as the continuity of modalities, m-saturation and point closure. Image compact modal frames also satisfy the Hennessy-Milner property.

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